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An approximate method for Abel inversion using Chebyshev polynomials

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ABSTRACT

Many problems in physics like reconstruction of the radially distributed emissivity from the line-of-sight projected intensity, the 3-D image reconstruction from cone-beam projections in computerized tomography, etc. lead naturally, in the case of radial symmetry, to the study of Abel's type integral equation. In this paper, a new stable algorithm based on shifted Chebyshev polynomial approximation is presented and analyzed. First, Chebyshev operational matrix of integration P is constructed and then it is used to reduce the integral equation to a system of algebraic equation which can be solved easily. The method is quite accurate and stable even though the approximations are performed using polynomials of degree up to 5. Some test examples from the plasma diagnostics are illustrated to demonstrate the effectiveness and stability of the proposed method.

1. Introduction

Abel's integral equation [1] occurs in many areas of physics and engineering such as plasma diagnostics, astronomy, geophysics and image analysis. Usually, physical quantities accessible to measurement are quite often related to physically important but experimentally inaccessible ones by Abel's integral equation. Some of the studies where Abel's integral equation is widely applicable are: seismology [2], satellite photometry of airglows [3], electron emission [4], atomic scattering [5], flame and plasma diagnostics [6], and X-ray radiography [7].

In flame and plasma diagnostics the Abel's integral equation relates the emission coefficient distribution function of optically thin cylindrically symmetric extended radiation source (particularly a plasma source) to the line-of-sight radiance measured in the laboratory. To obtain the physically relevant quantity from the measured one requires the inversion of the Abel's integral equation, and in case the object does not have radial symmetry, it requires, in principle, the inversion of Radon transform [8].

The relation between the radial distribution of the emission coefficient $\varepsilon(r)$ and measured intensity $I(y)$ from outside of the source is described by the Abel transform. The Abel transform can be interpreted as the projection of a circularly symmetric function along a set of parallel lines of sight which are at distance y from the origin (referring to the Fig. 1). Reconstruction of the emission coefficient from its projection is known as Abel inversion.

For a cylindrically symmetric, optically thin, extended radiation source the relationship between the emissivity $\varepsilon_\lambda(r)$ and the intensity $I_\lambda(y)$, as measured from outside the source is given as [9],

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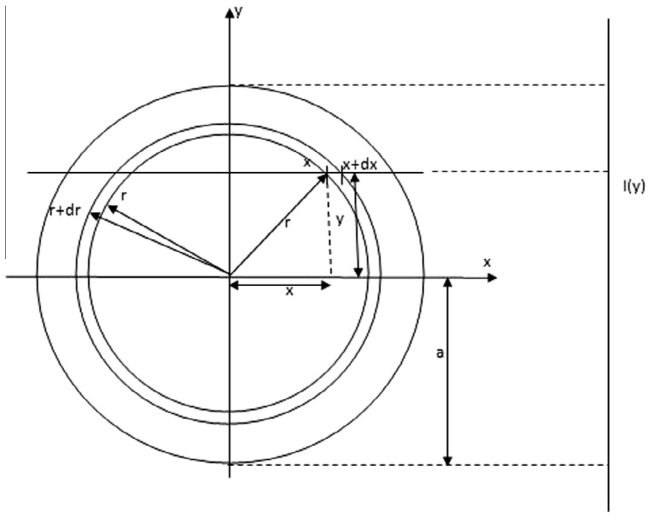

Fig. 1. Geometrical interpretation of the Abel transform in two dimensions with radius a.

$$I_\lambda(y) = \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} \varepsilon_\lambda(r)\, dr \tag{1}$$

for a particular wavelength λ, where y is the displacement of the intensity profile from the plasma centerline, r is the radial distance from the center of the source $x^2 + y^2 = r^2$, and a is the source radius. It is assumed that $\varepsilon_\lambda(r)$ vanishes for $r > a$, and hence $I_\lambda(y)$ vanishes for $|y| > a$. For simplicity, we take $a = 1.0$ in Eq. (1). Placing the variable of integration to r in Eq. (1), we obtain

$$I(y) = 2\int_y^1 \frac{\varepsilon(r)r}{\sqrt{r^2 - y^2}}\, dr, \quad 0 \leqslant y \leqslant 1, \tag{2}$$

a special form of Abel's integral equation, where we have dropped the suffix λ from $I_\lambda(y)$ and $\varepsilon_\lambda(r)$.

The analytical inversion of Eq. (2) is given as [10],

$$\varepsilon(r) = -\frac{1}{\pi}\int_r^1 \frac{1}{\sqrt{y^2 - r^2}} \frac{dI(y)}{d(y)}\, dy, \quad 0 \leqslant r \leqslant 1. \tag{3}$$

If the data function (projected intensity $I(y)$) is given approximately only at a discrete set of data points then the process of estimation of the solution function (emissivity $\varepsilon(r)$) becomes ill-posed because presence of small errors in the data $I(y)$ might cause large errors in the reconstructed solution. This is due to the fact that these formulae require differentiation of the measured data. In fact, two explicit analytic inversion formulae were given by Abel [1], but their direct application amplifies the experimental noise inherent in the radiance data significantly [11]. In 1982, a third analytic but derivative free inversion formula was obtained by Deutsch and Beniaminy [12] to avoid this problem.

In addition, some more numerical inversion methods [12–19] have been developed and each of these methods has some limitation depending upon the presence of error in the measured data. In 1992, Mejia et al. [20] have analyzed the stable Abel inversion through measured data on a discrete set of points using piecewise constant and piecewise linear interpolation techniques. Later, some new developments on Abel inversion have also been presented by many researchers such as Cho and Na [21], and George Chan and Hieftjen [22].

As per author's knowledge, the latest contributions on Abel's inversions are summarized as follows. In 2006, Yousefi [23] has provided Legendre wavelet based method for solving Abel integral equations. In [24], Pandey et al. have discussed analytical methods like Homotopy perturbation method (HPM), modified Homotopy perturbation method (MHPM), Adomian decomposition method (ADM) and modified Adomian decomposition method (MADM) for solving Abel integral equations. Further, Singh et al. [25], presented a stable algorithm for Abel's inversion using Bernstein's polynomials. Ma et al. [26–27], have presented Legendre polynomials and Legendre wavelets based stable algorithms for Abel's inversion. Li et al. [28], have provided and analyzed an efficient and stable method for Abel's inversion using generalized Taylor–Stieltjes polynomial approximation. Further, Huang et al. [29] discussed an approximate method for solving Abel integral equation by approximating the unknown function using Taylor series. In [30], Singh et al. constructed an operational matrix of integration based on orthonormal Bernstein polynomials, and used it to propose an algorithm for solving the Abel's integral equation. They have also studied the stability of the method under the effect of fixed noise. Consequently, the direct use of Eq. (3) is restricted and stable numerical methods are important.