Applied Mathematics and Computation 237 (2014) 730–751

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Revisited Simo algorithm for the plane stress state

Sanda Cleja-Ţigoiu [a,*], Nadia Elena Stoicuţa [a,b]

[a] University of Bucharest, Faculty of Mathematics and Computer Science, Academiei 14, 010014 Bucharest, Romania
[b] University of Petroşani, Department of Mathematics-Informatics, Institutului 20, 332006 Petrosani, Romania

ARTICLE INFO

Keywords:
Elasto-plastic model
Plane stress
Variational formulation
Finite element method
Trial state
Radial Return Mapping Algorithm
Numerical simulation

ABSTRACT

In this paper a new numerical approach of the elasto-plastic problem with mixed hardening in-plane stress state is proposed within the classical constitutive framework of small deformation. In the formalized problem, the non-zero normal component of the strain (namely, the component in the direction perpendicular to the plane of stress) is considered to be compatible with the plane stress. We eliminate the rate of strains which are developed normal to the plane and solve an appropriate plane problem in the strain setting. The integration algorithm for solving the elasto-plastic problem with mixed hardening realizes the coupling of Radial Return Algorithm, namely an update algorithm, with the finite element method applied to the discretized equilibrium balance equation. The solution of the pseudo-elastic nonlinear problem is then solved by Newton's method. The numerical algorithm (which is an incremental one) consists of the Radial Return Mapping Algorithm, originally proposed by Simo and Hughes (1998) adapted to the stress plane problem and coupled with a pseudo-elastic problem. We rebuild the algorithmic formula giving rise to the plastic factor in terms of the trial stress state and the algorithmic elasto-plastic tangent moduli which is requested to evaluate the Jacobian that is necessary to determine the displacement. The proposed algorithm has been named the revisited Simo algorithm and is applied to exemplify the mode of integration of the bi-dimensional problem.

1. Introduction

This paper deals with a new approach of the elasto-plastic problem with mixed hardening in the plane stress state, within the classical constitutive framework of small deformation. To solve the boundary and initial value problem, we use an iterative solution for the discretized equilibrium balance equation, a pseudo-elastic problem, which is coupled with the update algorithm required to solve the rate-type constitutive model.

Numerous references are devoted to elasto-plasticity with small strains with respect to the constitutive description, functional and numerical methods to solve the initial and boundary value problems. For the history and evolution of concepts that occur in the constitutive equations, as well as different theoretical formulations of elasto-plasticity, the reader is referred to Hill [11], Martin [19], Kachanov [7], Simo and Hughes [25] and Khan and Huang [8]. The evolution equations for tensorial hardening variables, which describe the mixed hardening, were proposed by Prager [22,23], Armstrong and Frederick [1], Chaboche [5]. Numerical tests are realized for elasto-plastic materials with kinematic hardening of Armstrong-Frederick type and scalar hardening function of Chaboche type and are compared with the experimental results

* Corresponding author.
E-mail addresses: tigoiu@fmi.unibuc.ro (S. Cleja-Ţigoiu), andnadia@yahoo.com (N.E. Stoicuţa).

http://dx.doi.org/10.1016/j.amc.2014.03.126
0096-3003/

obtained in laboratory by Broggiato et al. [4], using FEM code. The variation in time of the irreversible variables has been alternatively defined through the principle of the *maximum plastic dissipation*. This principle has been attributed by Hill [11] to Mises and considered, for instance, by Johnson [14], Lubliner [18] and Simo and Hughes [25]. This plays a basic role in the mathematical development of the problem in plasticity, see for instance Han and Reddy [10] and Duvaut and Lions [17]. The variational form of governing equations in plasticity in a discrete formalism was performed by Simo and Hughes [25] in Chapter 4, by assuming the coupling between the total free energy functional, the Lagrangian functional associated with the plastic dissipation, together with the Lagrange multiplier interpreted as plastic factor.

In Simo and Hughes [25], the authors pay attention to the so-called Radial Return Mapping Algorithm, which is considered to be the central problem in computational plasticity, and they mention that the method was first proposed by Wilkins [26] and Krieg and Key [16]. The trial stress is a basic concept of the Radial Return Mapping Algorithm and is computed from the elastic type constitutive equation for the frozen plastic behaviour at the previously reached value.

Within the J_2 plasticity theory the Radial Return Mapping Algorithm makes possible a closed-form solution of the problem, related to the rate-type description of constitutive models in elasto-plasticity. For an in-plane stress state, the algorithm can not be applied since the plane stress condition is violated. The procedure proposed by Simo and Hughes [25] follows Simo and Taylor [24] and performs the return mapping directly in the constrained plane stress space. Simo and Hughes [25] mentioned that the strain components $\varepsilon, \varepsilon^e$ and ε^p do not appear explicitly in their formulation since $\varepsilon^e_{33} = -\frac{v}{1-v}(\varepsilon^e_{11} + \varepsilon^e_{22})$, where v is the Poisson ratio, and $\varepsilon^p_{33} = -\varepsilon^p_{11} - \varepsilon^p_{22}$. Thus $\varepsilon_{33} = \varepsilon^e_{33} + \varepsilon^p_{33}$.

The Radial Return Mapping Algorithm in its various versions and extensions is based on the implicit backward Euler method, applied to the system of differential equations which describe the models. The algorithms and their accuracy and stability have been studied for instance by Hughes and Taylor [13] and Ortiz and Popov [21].

The finite element method (FEM) was generally used to solve variational equalities. The main ideas related to the FEM, which have been developed by Bathe [2], Belytschko et al. [3], Fish and Belytschko [9], Hughes [12], Johnson [15] and applied to various elastic problems, say for instance the isoparametric FEM, will be applied in this paper.

In Section 2, we formulate the differential type constitutive equations, which describe the behaviour of isotropic elasto-plastic material with isotropic and kinematic hardening. In the formalized rate-type constitutive model for in-plane stress, the non-zero normal component of the strain ε_{33} (namely, the component in the direction perpendicular to the plane of stress) has to be consistent with the requirement to have a plane stress state within the considered framework. We apply the procedure developed by Cleja-Ţigoiu [6] within the finite elasto-plasticity to eliminate the rate of strain which develops normal to the plane. We emphasize the peculiar aspects related to the generalized plane strain: (a) the matrix of elastic coefficients for isotropic material is replaced by the constitutive matrix of the elastic moduli $\mathcal{E}_{(2)}$, a fourth-order tensor associated with bi-dimensional symmetric tensors, Sym_2; (b) the normality of the flow rule to the yield surface is lost for the in-plane stress; (c) the new expression of the plastic factor is provided in terms of the plane strain rate.

In Section 3 the initial and boundary value problem, ***Problem P***, is formulated starting from the elastic type constitutive equation and the equilibrium equation to be satisfied by the stress, together with the rate-type constitutive description of the elasto-plastic material. To solve numerically ***Problem P***, we consider the discretized weak formulation associated with the equilibrium equations, namely a pseudo-elastic problem, in order to find the stress σ_{n+1} and the displacement vector u_{n+1} at time t_{n+1}.

In Section 4, for a given incremental strain history, the update algorithm is built to solve the differential-type system which describes the in-plane stress state. We use here the Radial Return Algorithm proposed by Simo and Hughes [25], in a revised form adapted to the in-plane stress state. We rebuild the algorithmic expression of the plastic factor in terms of the trial stress state and derive the new algorithmic elasto-plastic tangent moduli for the isotropic-kinematic hardening model.

Next, in Section 5 the FEM is used to solve the pseudo-elastic problem, which is reduced to a non-linear system for the displacement vector u_{n+1} at time t_{n+1}. The displacement u_{n+1} is then solved by Newton's method, while to evaluate the Jacobian of the system the previously calculated elasto-plastic moduli are requested.

To exemplify the numerical integration of the bi-dimensional problem and prove the efficiency of the numerical algorithm proposed in Section 6, we consider two examples in Section 7. In the numerical simulations a trapezoidal plate, with and without a hole, has been deformed for an in-plane stress generated by periodic forces applied on the horizontal edge of the plate. The numerical results of the simulations for the in-plane stress were performed using the material parameters given by Broggiato et al. [4] and compared to the solution that corresponds to Simo and Taylor [24], the so-called Simo solution.

Further we shall use the following notations:

$\mathbb{R}$ is the set of real numbers, and $\mathbb{R}_{\leqslant 0} = \{x \in \mathbb{R} | x \leqslant 0\}$;
Lin, Lin^+ are the sets of second-order tensors and the corresponding elements with positive determinant, respectively;
$Sym \subset Lin$ the set of symmetric tensors; Sym_2 the set of symmetric plane tensors;
$\sigma' = dev\sigma = \sigma - \frac{1}{3}(tr\sigma)I_3$ the deviatoric part of the stress tensor;
$\mathcal{E}$ the tensor of elastic moduli (the fourth order tensor); $\lambda_a,\ \mu_a$ the Lamé elastic constants;
E the Young modulus; v the Poisson ratio;
$\{e_i\}$ a Cartesian basis in $\mathbb{R}^3$; $I_2 = \delta_{ij} e_i \otimes e_j$ the second-order identity tensor in $\mathbb{R}^2$; I_3 the identity tensor in $\mathbb{R}^3$;
$I_4 = \frac{1}{2}\left[\delta_{ik}\delta_{jl} + \delta_{ij}\delta_{kl}\right] e_i \otimes e_j \otimes e_k \otimes e_l$ the fourth-order identity tensor;