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A general family of third order method for finding multiple roots ☆

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ABSTRACT

In this paper, we describe a general family of iterative methods for approximating a multiple root z with multiplicity m of a complex defined function. Almost of the family of the methods existing in the literature that use two-function and one-derivative evaluations are a special choice of this general method. We give some conditions to have the third order of convergence and we discuss how to choose a small asymptotic error constant which may be affect the speed of the convergence. Using Mathematica with its high precision compatibility, we present some numerical examples to confirm the theoretical results.

1. Introduction

Newton's method is one of the fundamental tools in numerical analysis, operations research, optimization and control. It has numerous applications in management science, industrial and financial research, data mining. Newton's method is a method for finding successively better approximations to the roots (or zeroes) of a real or complex function. The Newton's method in one variable has a second order of convergence (see Schröder [25], for instance) and it is implemented as follows: Given a function f defined over the real or complex x, and its derivative f', we begin with a first guess x_0 for a root of the function f. Provided the function is reasonably well-behaved a better approximation x_1 is

$$x_1 = x_0 - m\frac{f(x_0)}{f'(x_0)}.$$

where m is the multiplicity of the root.

There is a vast literature on the solution of nonlinear equation, see [18,23,28,31] and references therein. In recent years, some cubically convergent modified Newton's method for multiple roots have been proposed and analyzed, see for example [2,4,5,7,9–12,14,17,19,24,29]. These methods can be classified into two classes. The first one is based on the second derivative of the function f and the second one is based on the function f and its derivative f'. Several fourth order methods also are developed, see [8,15,16,20,21,30], for instance. All these fourth order methods require one evaluation of the function and at least two evaluations of its first derivative per iteration. The iterative methods for finding roots are classified by several

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criteria [28]: the order, informational efficiency and efficiency index. Recently in [22], Neta et al. consider other criteria, namely the basin of attraction of the method and its dependence on the order.

Our aim in this paper is to develop a method which use exactly in each iteration two evaluations of f and one evaluation of its derivative and we propose a general way to construct a cubically convergent iterative method for finding multiple roots. By our method we can find almost the existing methods in the literature which use two-function and one-derivative evaluations and construct novel other ones. Using the properties of the fixed point, we prove the third convergence of our method and we give the asymptotic error constant. We also discuss how to obtain a small possible asymptotic error constant which may affect the convergence speed of the proposed method.

The paper is organized as follows: In Section 2, we give some tools needed to prove the convergence of the proposed iterative method. In Section 3, we propose a general family of methods for finding multiple roots of a complex-valued function, starting with a reasonable initial guess. We also state the main result concerning the third order of convergence and asymptotic error constant. In Section 4, we give some special family of cubically convergent methods and we discuss how to obtain a small possible asymptotic error constant. Finally, in Section 5 we give some numerical examples to illustrate the theoretical results.

2. Preliminary

Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be a function having a multiple root z with multiplicity m and analytic in a neighborhood of this root. An alternative sufficient method for computing a solution of the equation $f(x) = 0$ is given by rewriting it in the equivalent form $g(x) - x = 0$, where g is a certain complex-valued function, defined and analytic in a neighborhood of z. Upon such a transformation, the problem of solving the equation $f(x) = 0$ is converted into one of finding the fixed point of g.

In order to approximate this fixed point, we choose an initial approximation $x_0 \in \mathbb{C}$ and generate the sequence $\{x_n\}_{n=0}^{\infty}$ according to the following recurrence relation:

$$x_{n+1} = g(x_n) \quad n \geqslant 0. \tag{2.1}$$

Let $r \in \mathbb{N}$ with $\mathbb{N}$ be the set of natural numbers. Assume that

$$\begin{cases} |g^{(r)}(z)| < 1, \text{ if } r = 1; \\ g^{(i)}(z) = 0 \text{ for } i \in \{1 \ldots r-1\} \quad \text{and } g^{(r)}(z) \neq 0, \text{ if } r \geqslant 2. \end{cases} \tag{2.2}$$

Suppose also that x_n belongs to a sufficiently small neighborhood of z for $n \geqslant 0$. By applying Taylor's theorem we obtain

$$x_{n+1} = g(x_n) = g(z) + \frac{g^{(r)}(\xi)(x_n - z)^r}{r!}, \tag{2.3}$$

where $\xi \in \mathcal{B}$ and $\mathcal{B}$ is a small ball with center z and containing x_n.

Since g is continuous at z, there exists, for all given $\varepsilon > 0$, a number δ such that

$$|x_{n+1} - z| = |g(x_n) - g(z)| = |g^{(r)}(\xi)|\left|\frac{(x_n - z)^{r-1}}{r!}\right||x_n - z| < \varepsilon, \tag{2.4}$$

for every x_n with $|x_n - z| < \delta$.

Let $\mathbf{J} = \{x \in \mathbb{C} / |x - z| < \delta\}$. Since $g^{(r)}$ is continuous on $\mathbf{J}$, there exists a strictly positive real M such that $|g^{(r)}(x)| \leqslant M, \ \forall x \in \mathbf{J}$. In the case $r = 1$, using Relation (2.2), we can take $M < 1$.

If we choose

$$\delta < \begin{cases} \varepsilon, & \text{if } r = 1; \\ \min\left(\varepsilon, (r!/M)^{1/(r-1)}\right), & \text{if } r \geqslant 2, \end{cases}$$

then $|x_{n+1} - z| = |g(x_n) - g(z)| < |x_n - z|$. Therefore $g(\mathbf{J}) \subset \mathbf{J}$.

From Relation (2.4), we obtain

$$|x_{n+1} - z| = |g(x_n) - g(z)| \leqslant K|x_n - z|, \tag{2.5}$$

where

$$K = \begin{cases} M, & \text{if } r = 1; \\ M\delta^{(r-1)}/r!, & \text{if } r \geqslant 2. \end{cases}$$

It is easy to see that the function g is a contraction on $\mathbf{J}$ for any r. Consequently g has a unique fixed point in $\mathbf{J}$. Moreover, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by Relation (2.1) converges to z as $n \to \infty$ for any starting value x_0 in $\mathbf{J}$ (see [1,3,6,13,26,27], for instance).

Put $e_n = |x_n - z|$, then for the method (2.1) the order of convergence is equal to r and the asymptotic error constant η (see [6,26,28] and references therein) is given by

$$\eta = \lim_{n\to\infty}\left|\frac{e_{n+1}}{e_n^r}\right| = \frac{|g^{(r)}(z)|}{r!}. \tag{2.6}$$