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Different anomalies in a Jarratt family of iterative root-finding methods ☆

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ABSTRACT

In this paper, the behavior of a Jarratt family of iterative methods applied to quadratic polynomials is studied. Some anomalies are found in this family be means of studying the dynamical behavior of this fourth-order family of methods. Parameter spaces are shown and the study of the stability of all the fixed points is presented. Dynamical planes for members with good and bad dynamical behavior are also provided.

1. Introduction

One of the most frequently problems in Sciences and more specifically in Mathematics is solving a nonlinear equation $f(z) = 0$, with $f : \mathbb{C} \to \mathbb{C}$. Except in very special cases, the solutions of this equations cannot be solved in a direct way. That is why most of the methods for solving these equations are iterative.

The dynamical properties related to an iterative method applied to polynomials give important information about its stability and reliability. In recently studies, authors such us Cordero et al. [9,10], Amat et al. [1–4], Gutiérrez et al. [12], Chun et al. [8] and many other ones [6,14,15] have found interesting dynamical planes, including periodical behavior and others anomalies. The main interest in this paper is the study of the parameter spaces associated to a family of iterative methods, which allow us to distinguish between the good and bad methods in terms of its numerical properties.

In this work, the fourth-order family of Jarratt iterative method introduced by Amat et al. in [1] whose iterative expression is

$$J(z) = z - u_f(z) + \frac{3}{4}u_f(z)h_f(z)\frac{1+\beta h_f(z)}{1+\left(\frac{3}{2}+\beta\right)h_f(z)}, \tag{1}$$

where $u_f(z) = \frac{f(z)}{f'(z)}$, $h_f(z) = \frac{f'(z-\frac{2}{3}u_f(z))-f'(z)}{f'(z)}$ and β is a complex parameter. Notice that every member of this family is an order four iterative method. As Amat et al. stated in [1] this family contains Jarratt's iterative map ($\beta = 0$) and the inverse-free Jarratt's map ($\beta = -3/2$). Moreover, they proved that this family of methods satisfies the Scaling Theorem. In this paper, the dynamics of this family applied to an arbitrary quadratic polynomial $p(z) = (z-a)(z-b)$ will be analyzed, characterizing the stability of all the fixed points. The graphic tool used to obtain the parameter space and the different dynamical planes have been introduced by the author in [13], but there exist other techniques such us the one given by Chicharro et al. in [7].

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Considering the conjugacy map given by Blanchard, in [5]

$$M(z) = \frac{z-a}{z-b}, \quad M^{-1}(z) = \frac{zb-a}{z-1}, \tag{2}$$

which obviates the roots, the family $J(z,\beta)$ is conjugated to

$$J(z,\beta) = \frac{3z^6 + z^4(3-8\beta) + z^5(6-4\beta)}{3 + z^2(3-8\beta) + z(6-4\beta)}. \tag{3}$$

The general convergence of the family of methods (1) for quadratic polynomials will be studied. The rest of the paper is organized as follows: in Section 2 some of the basic dynamical concepts related to the complex plane are presented, in Section 3 the stability of the fixed points of the family $J(z,\beta)$ is studied and in Section 4 the dynamical behavior of the family of Jarratt methods is analyzed, where the parameter space and some selected dynamical planes are presented. Finally, the conclusions drawn to this study are presented in the concluding Section 5.

2. Dynamical concepts of complex dynamics

In this Section, some dynamical concepts of complex dynamics that will be used in this work are shown. As the family of iterative method $J(z,\beta)$ applied to polynomials gives a rational function, the focus will be centered in them. Given a rational function $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, where $\hat{\mathbb{C}}$ is the Riemann sphere, the *orbit of a point* $z_0 \in \hat{\mathbb{C}}$ is defined as

$$\left\{z_0, R(z_0), R^2(z_0), \ldots, R^n(z_0), \ldots\right\}.$$

A point $z_0 \in \bar{\mathcal{C}}$, is called a *fixed point* of $R(z)$ if it verifies that $R(z) = z$. Moreover, z_0 is called a *periodic point* of period $p > 1$ if it is a point such that $R^p(z_0) = z_0$ but $R^k(z_0) \neq z_0$, for each $k < p$. Moreover, a point z_0 is called *pre-periodic* if it is not periodic but there exists a $k > 0$ such that $R^k(z_0)$ is periodic.

There exist different types of fixed points depending on its associated multiplier $|R'(z_0)|$. Taking the associated multiplier into account a fixed point z_0 is called:

- *superattractor* if $|R'(z_0)| = 0$,
- *attractor* if $|R'(z_0)| < 1$,
- *repulsor* if $|R'(z_0)| > 1$,
- and *parabolic* if $|R'(z_0)| = 1$.

The fixed points that do not correspond to the roots of the polynomial $p(z)$ are called *strange fixed points*. On the other hand, a *critical point* z_0 is a point which satisfies that, $R'(z_0) = 0$.

The basin of attraction of an attractor α is defined as

$$\mathcal{A}(\alpha) = \{z_0 \in \hat{\mathbb{C}} : \ R^n(z_0) \to \alpha, \ n \to \infty\}.$$

The *Fatou set* of the rational function R, $\mathcal{F}(R)$, is the set of points $z \in \hat{\mathbb{C}}$ whose orbits tend to an attractor (fixed point, periodic orbit or infinity). Its complement in $\hat{\mathbb{C}}$ is the *Julia set*, $\mathcal{J}(R)$. That means that the basin of attraction of any fixed point belongs to the Fatou set and the boundaries of these basins of attraction belong to the Julia set.

3. Study of the fixed points and its stability

It is clear that $z = 0$ and $z = \infty$ are fixed points (related to the roots a and b, respectively, of the polynomial $p(z)$). On the other hand, we see that $z = 1$ is a strange fixed point, which is associated with the original convergence to infinity. Moreover, there are also another four strange fixed points which correspond with the roots of the polynomial

$$q(z) = 3 + 9z + 12z^2 + 9z^3 + 3z^4 - 4z\beta - 12z^2\beta - 4z^3\beta,$$

whose analytical expression, depending on β, are:

$$ex_1(\beta) = \frac{1}{12}\left(-9 + 4\beta - \sqrt{9 + 8\beta(9+2\beta)} - \sqrt{32\beta^2 - 8\beta\sqrt{9+8\beta(9+2\beta)} + 18\left(-3 + \sqrt{9+8\beta(9+2\beta)}\right)}\right);$$

$$ex_2(\beta) = \frac{1}{12}\left(-9 + 4\beta - \sqrt{9 + 8\beta(9+2\beta)} + \sqrt{32\beta^2 - 8\beta\sqrt{9+8\beta(9+2\beta)} + 18\left(-3 + \sqrt{9+8\beta(9+2\beta)}\right)}\right);$$

$$ex_3(\beta) = \frac{1}{12}\left(-9 + 4\beta + \sqrt{9 + 8\beta(9+2\beta)} - \sqrt{-18\left(3 + \sqrt{9+8\beta(9+2\beta)}\right) + 8\beta\left(4\beta + \sqrt{9+8\beta(9+2\beta)}\right)}\right);$$