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The minimal rank of $A - BX$ with respect to Hermitian matrix ☆

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ABSTRACT

In this paper, we discuss the minimal rank of $A - BX$ when X is Hermitian by applying singular value decomposition and some rank equalities of matrices, and obtain a representation of the minimal rank. Based on the representation, we obtain necessary and sufficient conditions for the matrix equation $BX = A$ to have Hermitian solutions.

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1. Introduction

Throughout this paper, we adopt the following notation. The symbol $\mathbb{C}^{m\times n}$ is the set of all $m \times n$ complex matrices, $\mathbb{H}^{m\times m}$ ($\mathbb{U}^{m\times m}$) is the set of all $m \times m$ Hermitian matrices (unitary matrices). The *conjugate transpose* of A is denoted by A^H. The *inertia* of a Hermitian matrix A is defined to be the triplet $\ln(A) = \{n_+(A), n_-(A), n_0(A)\}$, where $n_+(A)$, $n_-(A)$ and $n_0(A)$ are the numbers of the positive, negative and zero eigenvalues of A counted with multiplicities, respectively. The two numbers $n_+(A)$ and $n_-(A)$ are called the positive and negative index of inertia, respectively. The symbol I_p denotes the p-by-p identity matrix. The row block matrix consisting of A and B is denoted by $[A \quad B]$ and its rank is denoted by $\mathrm{rank}([A \quad B])$. The symbols $n_+\{A,B\}$ and $n_-\{A,B\}$ denote the positive and negative index of the inertia of the block Hermitian matrix

$$\begin{bmatrix} i\left(A - A^H\right) & B^H \\ B & 0 \end{bmatrix},$$

respectively. The *Moore–Penrose inverse* of $A \in \mathbb{C}^{m\times n}$ is defined as the unique $X \in \mathbb{C}^{n\times m}$ satisfying

$$(1) \quad AXA = A, \ \ (2) \ XAX = X, \ \ (3) \ (AX)^H = AX, \ \ (4) \ (XA)^H = XA,$$

and is usually denoted by $X = A^\dagger$ (see [1]). Furthermore, the symbols E_A and F_A stand for the two orthogonal projections:

$$E_A = I_m - AA^\dagger \ \text{ and } \ F_A = I_n - A^\dagger A.$$

Consider the following well-known linear matrix equation

$$BX = A, \tag{1.1}$$

where $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{m\times n}$ are given matrices. It is obviously that (1.1) has a Hermitian solution if and only if the minimal rank of $A - BX$ with respect to $X = X^H$ equals 0. The problem for algebraic properties of Hermitian solutions of (1.1) has been widely studied. Khatri and Mitra[7] deduced a necessary and sufficient condition for the existence of a Hermitian solution to (1.1). They show that a Hermitian solution X exists if and only if

$$BB^\dagger A = A \ \text{ and } \ BA^H = AB^H. \tag{1.2}$$

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Horn et al. [6] studied (1.1) in a given *congruence class. Li et al.[8] studied the submatrices in a solution X of (1.1) and given some necessary and sufficient conditions for the existence of solutions to be local positive (negative) semidefinite by elementary block matrix operations. Guo and Huang[5] studied the problem for determining the extremal ranks of one matrix expression with respect to Hermitian matrix by applying the quotient singular value decomposition, especially when A is a skew-Hermitian matrix,

$$\min_{X=X^H} \operatorname{rank}(A-X) = \max\{n_+(iA), n_-(iA)\}. \tag{1.3}$$

It is well known that rank of matrix is an important tool in matrix theory and its applications, and many problems are closely related to the ranks of some matrix expressions under some restrictions. Recently, the extremal ranks of some matrix expressions have found many applications in control theory, statistics and economics[2,3,9], etc., and hence the problems for determining the extremal ranks of matrix expressions have been widely studied [4,10,11,13–17],etc. In this paper, we will study the minimal rank of $A-BX$ with respect to $X=X^H$ by applying singular value decomposition (SVD), Sylvester's law of inertia and some rank equalities of matrices.

2. Preliminaries

In order to find the minimal rank of $A-BX$ with respect to Hermitian matrix, we need the following results on minimal ranks of two special matrix expressions.

Denote

$$\mathcal{F}(\mathcal{X},\mathcal{Y}) = \mathcal{H} - \mathcal{X} - \mathcal{Y}\mathcal{J}.$$

In the following subsection, we will study the minimal rank of $\mathcal{F}(\mathcal{X},\mathcal{Y})$ when $\mathcal{X}=\mathcal{X}^H$ and $\mathcal{Y}$ are arbitrary matrices.

2.1. The minimal rank of $\mathcal{F}(\mathcal{X},\mathcal{Y})$ when $\mathcal{X}=\mathcal{X}^H$ and $\mathcal{Y}$ are arbitrary matrices

Suppose that the matrix $\mathcal{H}\in\mathbb{C}^{m_1\times m_1}$ and $\mathcal{J}\in\mathbb{C}^{l_1\times m_1}$ are given. Write

$$\mathcal{H}_H = \frac{\mathcal{H}+\mathcal{H}^H}{2}, \quad \mathcal{H}_S = \frac{\mathcal{H}-\mathcal{H}^H}{2},$$

then

$$\mathcal{H} = \mathcal{H}_H + \mathcal{H}_S,$$

where $\mathcal{H}_H$ is an $m_1\times m_1$ Hermitian matrix, and $\mathcal{H}_S$ is an $m_1\times m_1$ skew-Hermitian matrix. Notice that for any $\mathcal{X}\in\mathbb{H}^{m_1\times m_1}$

$$\min_{\mathcal{X}=\mathcal{X}^H}\operatorname{rank}(\mathcal{H}-\mathcal{X}) = \min_{\mathcal{X}=\mathcal{X}^H}\operatorname{rank}(\mathcal{H}_S-\mathcal{X}),$$

therefore, it follows from applying (1.3) that

$$\min_{\mathcal{X}=\mathcal{X}^H}\operatorname{rank}(\mathcal{H}-\mathcal{X}) = \max\left\{n_+\left(i(\mathcal{H}-\mathcal{H}^H)\right), n_-\left(i(\mathcal{H}-\mathcal{H}^H)\right)\right\}. \tag{2.1}$$

The following formula for inertias of block matrix can be derived from Theorem 2.3 [12]

$$n_\pm\{\mathcal{H},\mathcal{J}\} = \operatorname{rank}(\mathcal{J}) + n_\pm\left(F_{\mathcal{J}}\left(i(\mathcal{H}-\mathcal{H}^H)\right)F_{\mathcal{J}}\right). \tag{2.2}$$

We have the following result.

Lemma 2.1. *Suppose that $\mathcal{H}\in\mathbb{C}^{m_1\times m_1}$ and $\mathcal{J}\in\mathbb{C}^{l_1\times m_1}$ are given, and $\mathcal{X}\in\mathbb{H}^{m_1\times m_1}$ and $\mathcal{Y}\in\mathbb{C}^{m_1\times l_1}$ are arbitrary matrices. Then the minimal rank of the matrix expression $\mathcal{H}-\mathcal{X}-\mathcal{Y}\mathcal{J}$ is*

$$\min_{\mathcal{X}=\mathcal{X}^H,\mathcal{Y}}\operatorname{rank}(\mathcal{H}-\mathcal{X}-\mathcal{Y}\mathcal{J}) = \max\{n_+\{\mathcal{H},\mathcal{J}\}, n_-\{\mathcal{H},\mathcal{J}\}\} - \operatorname{rank}(\mathcal{J}). \tag{2.3}$$

Proof. Let the SVD of $\mathcal{J}$ be

$$\mathcal{J} = \mathcal{U}\begin{bmatrix}\widehat{\Sigma} & 0\\ 0 & 0\end{bmatrix}\mathcal{V}^H,$$

and partition

$$\mathcal{V}^H\mathcal{H}\mathcal{V} = \begin{bmatrix}\mathcal{H}_1 & \mathcal{H}_2\\ \mathcal{H}_3 & \mathcal{H}_4\end{bmatrix},$$

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