Applied Mathematics and Computation 233 (2014) 72–84

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Variational minimization on string-rearrangement surfaces, illustrated by an analysis of the bilinear interpolation

Daud Ahmad [a,*], Bilal Masud [b]

[a] Department of Mathematics, University of the Punjab, Lahore, Pakistan
[b] Center for High Energy Physics, University of the Punjab, Lahore, Pakistan

ARTICLE INFO

Keywords:
Coons patch
String-rearrangement
Bilinear interpolation
Variational improvement
Minimal surfaces

ABSTRACT

In this paper we present an algorithm to reduce the area of a surface spanned by a finite number of boundary curves by initiating a variational improvement in the surface. The ansatz we suggest consists of original surface plus a variational parameter t multiplying the numerator H_0 of mean curvature function defined over the surface. We point out that the integral of the square of the mean curvature with respect to the surface parameter becomes a polynomial in this variational parameter. Finding a zero, if there is any, of this polynomial would end up at the same (minimal) surface as obtained by minimizing more complicated area functional itself. We have instead minimized this polynomial. Moreover, our minimization is restricted to a search in the class of all surfaces allowed by our ansatz. All in all, we have not yet obtained the exact minimal but we do reduce the area for the same fixed boundary. This reduction is significant for a surface (hemiellipsoid) for which we know the exact minimal surface. But for the bilinear interpolation spanned by four bounding straight lines, which can model the initial and final configurations of re-arranging strings, the decrease remains less than 0.8 percent of the original area. This may suggest that bilinear interpolation is already a near minimal surface.

1. Introduction

Variational methods are one of the active research areas of the optimization theory [1]. A variational method tries to find the best values of the parameters in a trial function that optimize, subject to some algebraic, integral or differential constraints, a quantity dependant on the ansatz. A simple example of such a problem may be to find the curve of shortest length connecting two points. The solution is a straight line between the points in case of no constraints and simplest metric, otherwise possibly many solutions may exist depending on the nature of constraints. Such solutions are called geodesics [2–4]. One of the related problems is finding the path of stationary optical length connecting two points, as the Fermat's principle says that the rays of light traverse such a path. Another related problem is a *Plateau problem* [5,6] which is finding the surface with minimal area enclosed by a given curve. This problem is named after the blind Belgian physicist Joseph Plateau, who demonstrated in 1849 that a minimal surface can be obtained by immersing a wire frame, representing the boundaries, into soapy water. The Plateau problem attracted mathematicians like Schwarz [7] (who discovered D (diamond), P (primitive), H (hexagonal), T (tetragonal) and CLP (crossed layers of parallels) triply periodic surfaces), Riemann [5], and Weierstrass [5].

* Corresponding author.
E-mail addresses: daudahmadpu@yahoo.com (D. Ahmad), bilalmasud.chep@pu.edu.pk (B. Masud).

http://dx.doi.org/10.1016/j.amc.2014.01.172

Although mathematical solutions for specific boundaries had been obtained for years, but it was not until 1931 that the American mathematician Jesse Douglas [8] and the Hungarian Tibor Radó [9] independently proved the existence of a minimal solution for a given simple closed curve. Their methods were quite different. Douglas [8] minimized a functional now named as Douglas–Dirichlet Integral. This is easier to manage but has the same extremals in an unrestricted search [10] as the area functional. Douglas results held for arbitrary simple closed curve, while Radó [9] minimized the energy. The work of Radó was built on the previous work of Garnier [11] and held only for rectifiable simple closed curves. Many results were obtained in subsequent years, including revolutionary achievements of Tonelli [12], Courant [13,14], Morrey [15,16], McShane [17], Shiffman [18], Morse [19], Tompkins [19], Osserman [20], Gulliver [21] and Karcher [22] and others.

In addition to finding (above mentioned) alternative functionals, the search can be limited to a certain class of surfaces. A widely used such restriction is to search among all Bézier surfaces with the given boundary. Bézier models are widely used in computer aided geometric design (CAGD) because of their suitable geometric properties. For a control net $\mathbf{P}_{ij}$ of a two dimensional parametric Bézier surface is given by

$$\mathbf{x}(u,v)=\sum_{i=0}^{n}\sum_{j=0}^{m}B_i^n(u)B_j^m(v)\mathbf{P}_{ij}, \tag{1}$$

where u, v are the parameters, $B_i^n(u)=\binom{n}{i}u^i(1-u)^{n-i}$, the Bernstein polynomials of degree n and $\binom{n}{i}=\frac{n!}{i!(n-!)}$, binomial coefficients and $D=[0,1]\times[0,1]$. The minimal Bézier surfaces as an example of the extremal of discrete version of Dirichlet functional may be found in the Monterde work [10], a restricted Plateau-Bézier problem defined as the surface of minimal area among all Bézier surfaces with the given boundary. A use of Dirichlet method and the extended bending energy method to obtain an approximate solution of Plateau-Bézier problem may be seen in work by Chen et al. [23]. Another restriction may be to find a surface in the parametric polynomial form as it can be seen in the Ref. [24] that finds a class of quintic parametric polynomial minimal surfaces. Bézier surfaces exactly deal with the case that the prescribed borders are polynomial curves. A more general case of borders is taken in Ref. [25] that study the Plateau-quasi-Bézier problem which includes the case when the boundary curves are catenaries and circular arcs. The Plateau-quasi-Bézier problem is related to the quasi-Bézier surface with minimal area among all the quasi-Bézier surfaces with prescribed border. They minimize the Dirichlet functional in place of original area functional.

An emerging use of minimal surfaces in physics is that in string theories. A classical particle travels a geodesic with least distance whereas a classical string is an entity which traverses a minimal area. Amongst the string theories used in physics, two are worth mentioning. One is the theory of quantum chromodynamics (QCD) strings that model the gluonic field confining a quark and an antiquark within a meson. (The gluonic field connecting three quarks, within a proton or neutron, is modeled through Y-shaped strings. For a system composed of more than three quarks, minimization of the total length of a string network with only Y-shaped junctions may be a non-trivial Steiner-Tree Problem [26]). In the other string theory (or theories) string vibrations are supposed to generate different elementary particles of the present high energy physics. Quite often string theories need a surface spanning the boundary composed of curves either connecting particles or describing the time evolution of particles. An important case can be a fixed boundary composed of four external curves. A common application of this boundary can be the time evolution of a string parameterized by σ or β [27] variable; the time evolution itself is parameterized by the symbol τ, the proper time of relativity. In this case two bounding curves parameterized by the respective σ or β represent the initial and final configurations of a string, and the other two curves (parameterized by the respective τ variables) describe the time evolution of the two ends of a string.

String theories take action to be proportional to area. Combining this with the classical mechanics demand of the least action, minimal surfaces spanning the corresponding fixed boundaries get their importance. For example, see Eq. (13) of Ref. [27] for the Nambu-Goto ansatz for the minimal surface area and compare it with Eqs. (14) and (15) below, along with Ref. [28] for Nambu-Goto strings. Also relevant is the use in Ref. [29] of Wilson minimal area law (MAL) to derive the quark antiquark potential in a certain approximation. A surface spanned by such a boundary is in space–time of relativity. An ordinary 3-dimensional spatial surface can span a boundary composed of two 3-dimensional curves connecting four particles and two other curves connecting the same four particles in a re-arranged (or exchanged) clustering; see for example Fig. 2 of Ref. [30] and Fig. 5 of Ref. [31]. An explicit expression of such a spanning surface can be found in Eqs. (3) and (4) of Ref. [32] and Eq. (22) of Ref. [33]. This is a bilinear interpolation in ordinary 3-dimensional space and is similar to the linear interpolations in above mentioned Eq. (13) of Ref. [27], Eq. 4.7 of Ref. [34] and Eq. 3.4 of Ref. [29]. Ref. [34] clarifies that such a surface is used as a *replacement* to the exact minimal surfaces for the corresponding boundaries; see Section 2 below for a minimal surface in the differential geometry. Even non-minimal surfaces have some usage in the mathematical modeling of quantum strings because (1) in contrast to classical strings, quantum strings can have any action and hence area as described by the path integral version of the quantum mechanics (see Eq. (1) of [35]) and (2) any surface spanning a boundary composed of quark lines (or quark connecting lines) corresponds to a physically allowed (gauge invariant) configuration of the gluonic field between these quarks; compare the non-minimal surface of Fig. 10.5 of Ref. [36] with the minimal surface for the same boundary in Fig. 10.1 of the same Ref. [36]. But it cannot be denied that minimal surfaces are the most important of the spanning surfaces even in quantum theories. For example, the relation in Eq. 1.14 of Ref. [37] between an area and an important quantity (termed *Wilson loop*) related to the potential between a quark and antiquark connected by a QCD (gluonic) string is valid only if the area is of the minimal surface. (Though above mentioned Eq. (1) of Ref. [35] relates the Wilson