Applied Mathematics and Computation 233 (2014) 103–108

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On some linear combinations of commuting involutive and idempotent matrices

Marina Tošić

University of Niš, Faculty of Sciences and Mathematics, Serbia

ARTICLE INFO

Keywords:
Involutive matrix
k-Potent matrix
Idempotent matrix
Linear combination

ABSTRACT

We consider the problem of characterizing situations where a linear combination of commuting involutive matrices is a k-potent matrix, where k is a positive integer such that $k \geqslant 2$. Also, we consider the problem of characterizing situations where a linear combination of the form $c_1 I_n + c_2 A + c_3 B$ is a nonsingular or an involutive matrix, when idempotent matrices $A, B \in \mathbb{C}^{n\times n}$ satisfy different conditions and $c_1, c_2, c_3 \in \mathbb{C} \setminus \{0\}$.

1. Introduction

Let $\mathbb{C}^{n\times n}$ denote the set of all $n \times n$ complex matrices. By $\mathbb{C}_r^{n\times n}$ we will denote the set of all matrices from $\mathbb{C}^{n\times n}$ with a rank r. Also, we will use the following notation: for $k \in \mathbb{N}$ and $k > 1$, the set of complex roots of 1 shall be denoted by σ_k and if we set $\omega_k = e^{2\pi i/k}$ then $\sigma_k = \{\omega_k^0, \omega_k^1, \ldots, \omega_k^{k-1}\}$. $\oplus$ denotes a direct sum. We say that k and l are congruent modulo m, and we use the notation $k \equiv_m l$, if $m|(k-l)$. I_n will denote the identity matrix of order n. Also, recall that a matrix A is involutive if $A^2 = I_n$, tripotent if $A^3 = A$, essentially tripotent if $A^3 = A$ with $A^2 \neq \pm A$, k-potent if $A^k = A$, projector (an idempotent matrix) if $A^2 = A$, generalized projector if $A^2 = A^*$ and hypergeneralized projector if $A^2 = A^\dagger$. We use the notation C_n^P for the subsets of $\mathbb{C}^{n\times n}$ consisting of projectors (idempotent matrices), i.e.,

$$C_n^P = \{A \in \mathbb{C}^{n\times n} : A^2 = A\}.$$

For the cases: (i) A and B are idempotent, (ii) A is idempotent and B is tripotent, and (iii) A is idempotent and B is t-potent where A and B are commuting matrices, the problem of characterizing some and even all situations where a linear combination of the form $c_1 A + c_2 B$ is an idempotent matrix were studied in [1,5,2,4]. Idempotency of linear combinations of three idempotent matrices was studied in [3,5], while idempotency and tripotency of linear combinations of two commuting tripotent matrices was investigate in [6].

The problem of characterizing all situations where a linear combination of involutive matrices is a tripotent, an idempotent or an involutive matrix were studied in [7]. In this paper, we characterize situations where a linear combination of commuting involutive matrices is a k-potent, where $k \in \mathbb{N}$ with $k \geqslant 2$.

Also, the authors in [7] determined all situations where a linear combination of the form $c_1 A + c_2 B$ is an involutive matrix when A and B are idempotent or tripotent matrices. We give the solution to the problem when a linear combination of the form $c_1 I_n + c_2 A + c_3 B$ is a nonsingular or an involutive matrix, when idempotent matrices A and B satisfy one of the following conditions: $A - B = 0$ or $AB = B$ and $BA = A$ or $(A-B)^2 = A - B$ or $(A+B)^2 = A + B$. Also, we consider if it is possible that a

E-mail address: marinatosic@ymail.com

http://dx.doi.org/10.1016/j.amc.2014.01.110
0096-3003/

linear combination of the form $c_1I_n + c_2A + c_3B$ is an involutive matrix, when idempotent matrices A and B satisfy $ABA = BAB$ or $AB = BA$.

The matrices observed in the above mentioned papers and in this paper have an important role in the applied sciences, especially in statistical theory. More about this can be found in [2,7].

2. On linear combinations of commuting involutive matrices

We consider the problem of k-potency of a linear combination

$$c_1A + c_2B, \tag{1}$$

where $A,\ B$ are involutive matrices that commute, $k \in \mathbb{N},\ k \geqslant 2$ and $c_1, c_2 \in \mathbb{C} \setminus \{0\}$. We will divide our discussion into two cases: when k is odd and when k is even number.

Remark that if A and B are involutive matrices and $A = \lambda B$, then $A = B$ or $A = -B$.

Also, if $A = B$, then $c_1A + c_2B$ is k-potent, where k is odd if and only if $c_1 = -c_2$ or if there is $t \in \sigma_{k-1}$ such that $c_1 + c_2 = t$. If $A = -B$, then $c_1A + c_2B$ is k-potent, where k is odd if and only if $c_1 = c_2$ or if there is $t \in \sigma_{k-1}$ such that $c_1 - c_2 = t$. If k is even, then for $A = B$ the matrix $c_1A + c_2B$ is k-potent if and only if $c_1 = -c_2$ or $A = (c_1 + c_2)^{k-1}I$. If $A = -B$, then $c_1A + c_2B$ is k-potent, where k is even if and only if $c_1 = c_2$ or $A = (c_1 - c_2)^{k-1}I$. In particular, if $A = B = I$, then $c_1A + c_2B$ is k-potent if and only if $c_1 = -c_2$ or if there is $t \in \sigma_{k-1}$ such that $c_1 + c_2 = t$. Also, if $A = -B = I$, then $c_1A + c_2B$ is k-potent if and only if $c_1 = c_2$ or if there is $t \in \sigma_{k-1}$ such that $c_1 - c_2 = t$. If $A = B = -I$, then $c_1A + c_2B$ is k-potent if and only if $c_1 = -c_2$ or if there is $t \in \sigma_{k-1}$ such that $c_1 + c_2 = -t$.

First, we will state the following theorem.

Theorem 2.1. *Let $\{A_i\}_{i=1}^m$ be a finite commuting family of involutive matrices and $k_1, k_2, \ldots, k_m \in \mathbb{N}$ such that $\prod_{i=1}^m A_i^{k_i} \neq \pm I$ and let $c_1, c_2 \in \mathbb{C} \setminus \{0\},\ k \in \mathbb{N},\ k \geqslant 2$. Then a linear combination $c_1I + c_2\prod_{i=1}^m A_i^{k_i}$ is a k-potent matrix if and only if one of the following conditions is satisfied:*

(i) $c_1 = c_2 = \frac{1}{2}t,\ t \in \sigma_{k-1}$,
(ii) $c_1 = -c_2 = \frac{1}{2}t,\ t \in \sigma_{k-1}$,
(iii) *there exist $t_1, t_2 \in \sigma_{k-1},\ t_1 \neq \pm t_2$ such that $c_1 = \frac{t_1+t_2}{2}$ and $c_2 = \frac{t_1-t_2}{2}$.*

Proof. Since for an arbitrary finite commuting family of involutive matrices $\{A_i\}_{i=1}^m$ and arbitrary positive integer $k_1, k_2, \ldots, k_m$, $\prod_{i=1}^m A_i^{k_i}$ is still an involutive matrix, then $\prod_{i=1}^m A_i^{k_i}$ is diagonalizable and we can represent it by $\prod_{i=1}^m A_i^{k_i} = S(I \oplus -I)S^{-1}$, for a nonsingular matrix S. Now, $c_1I + c_2\prod_{i=1}^m A_i^{k_i}$ is a k-potent if and only if $c_1(I \oplus I) + c_2(I \oplus -I)$ is k-potent which is equivalent to $c_1 + c_2 \in \{0\} \cup \sigma_{k-1}$ and $c_1 - c_2 \in \{0\} \cup \sigma_{k-1}$. Since $c_1, c_2 \in \mathbb{C} \setminus \{0\}$, it immediately follows that c_1 and c_2 satisfy one of the conditions (i), (ii) and (iii). □

The following corollary presents when a linear combination $c_1I + c_2\prod_{i=1}^m A_i^{k_i}$ is idempotent, where $\{A_i\}_{i=1}^m$ is a finite commuting family of involutive matrices.

Corollary 2.2. *Let $\{A_i\}_{i=1}^m$ be a finite commuting family of involutive matrices and $k_1, k_2, \ldots, k_m \in \mathbb{N}$ such that $\prod_{i=1}^m A_i^{k_i} \neq \pm I$ and let $c_1, c_2 \in \mathbb{C} \setminus \{0\}, k \in \mathbb{N}, k \geqslant 2$. Then a linear combination $c_1I + c_2\prod_{i=1}^m A_i^{k_i}$ is idempotent if and only if*

$$(c_1, c_2) \in \left\{ \left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, -\frac{1}{2}\right) \right\}.$$

The following theorems present necessary and sufficient conditions for k-potency of the linear combination (1) when k is odd and when k is even number. First, we give necessary and sufficient conditions for k-potency of the linear combination (1) in the case when $k \in \mathbb{N},\ k \geqslant 2$ and $k \equiv_2 1$.

Theorem 2.3. *Let $A, B \in \mathbb{C}^{n\times n}$ be commuting involutive matrices such that $A \neq \pm B$ and let $c_1, c_2 \in \mathbb{C} \setminus \{0\},\ k \in \mathbb{N},\ k \geqslant 2$ and $k \equiv_2 1$. Then a linear combination $T = c_1A + c_2B$ is a k-potent matrix if and only if one of the following conditions is satisfied:*

(i) $c_1 = c_2 = \frac{1}{2}t,\ t \in \sigma_{k-1}$,
(ii) $c_1 = -c_2 = \frac{1}{2}t,\ t \in \sigma_{k-1}$,
(iii) *there exist $t_1, t_2 \in \sigma_{k-1},\ t_1 \neq \pm t_2$ such that $c_1 = \frac{t_1+t_2}{2}$ and $c_2 = \frac{t_1-t_2}{2}$.*

Proof. Multiplying $T^k = T$ by the nonsingular matrix A, we get that T is a k-potent matrix if and only if $c_1I + c_2AB$ is a k-potent matrix. Now, the proof follows from Theorem 2.1. □

As a corollary we get the following result from [7]: