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The isometric identities and inversion formulas of complex continuous wavelet transforms

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ABSTRACT

The reproducing kernel function of the image space of a family of complex wavelet transforms is presented. An admissible wavelet is obtained by convolution computation. Next, the correlative characterisation of the image space of the family of complex wavelet transforms is provided when the scale is fixed. Furthermore, the isometric identities and inversion formulas are obtained, which provide a theoretic basis for investigating the image space of the general wavelet transform.

1. Introduction

Wavelet analysis has been rapidly developed in the mathematical field. This analysis is a brilliant combination of pure mathematics and applied mathematics following the Fourier transform, which has been called a mathematics microscope. In addition, wavelet analysis is also one of the most obvious achievements of harmonic analysis in the past half century. Wavelet analysis can effectively provide useful information from signals, and it also can solve many difficult problems that cannot be solved using the Fourier transform. Currently, this analysis has made most of the important valuable achievements in, for instance, the field of signal analysis (see Ref. [1]), image processing (see Ref. [2]), sound processing (see Ref. [3]), and solutions of equations (see Refs. [4–7]). We all know that there is redundancy of information with wavelet transforms, and the wavelet transform coefficient has relevance in the wavelet transform image plane. We can also see that the reproducing the kernel Hilbert space is the basis of the wavelet transform, and the relevance region's magnitude is given by the reproducing the kernel. In addition, we can prove that its magnitude will decrease if the scale decreases. Therefore, we can conclude that the reproducing the kernel space plays an important role in the reconstruction of the continuous wavelet transform (see Refs. [8–11]). With the development of wavelet analysis, reproducing kernel theory has attracted increasing attention from many scholars. For example, Saitoh described reproducing kernels of the direct product of two Hilbert spaces (see Refs. [12–14]). Deng and Du have described the image space of wavelet transform such as that of Shannon, Meyer and Littlewood-Paley (see Refs. [15–17]). Li et al. further studied the image space of the complex Gauss wavelet transform (see Ref. [18]). Castro et al. have identified a general discretisation method for solving wide classes of mathematical problems by applying the theory of reproducing kernels (see Refs. [19,20]). It can be observed that continuous wavelet transform is the foundation of the reproducing kernel function of the image space. Therefore, we can choose the most suitable wavelet bases according to the structure of the reproducing kernel, which is based on the information redundancy of the continuous wavelet transform. In this paper, we present a family of admissible complex wavelets and describe the spaces with the help of

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Fock kernel. The paper demonstrates isometry and inversion formulas under the fixed scale-factor conditions and further provides a theoretical foundation for the studies of the general wavelet transform. It is obvious that the results of Ref. [18] are a special case for this article.

2. Definition and properties

In this section, we introduce the concept and properties involved with reproducing kernels and wavelets.

Definition 2.1. Let H be a Hilbert function space. Its elements are real-valued or complex-valued functions on an abstract set E. The inner can be denoted by

$$\langle f, g\rangle = \langle f(\cdot), g(\cdot)\rangle, \quad f, g \in H$$

for any fixed $q \in E, K(p,q) \in H$ as a function in p, and for any $f \in H$, and $q \in E$, we have

$$f(q) = \langle f(p), K(p,q)\rangle,$$

then $K(p,q)$ will be called the reproducing kernel of the Hilbert function space H, H will be called the reproducing kernel Hilbert space (RKHS).

Furthermore, the function $K(p,q)$ is uniquely determined by RKHS H. Therefore, it can be written as H_K, that is, $H = H_K$. Furthermore, we can also write the Definition below for the reproducing kernel via linear transformation.

Proposition 2.1. *Let $\mathcal{F}(E)$ be a linear space comprising all complex-valued functions on an abstract set E. Let $\mathcal{H}$ be a Hilbert space equipped with product $\langle .,. \rangle_{\mathcal{H}}$. Let h; $E \to \mathcal{H}$ be a Hilbert space $\mathcal{H}$-valued function on E. We consider the linear mapping which is $\mathcal{L}$ from $\mathcal{H}$into $\mathcal{F}(E)$as defined by*

$$f(p) = (\mathcal{L}\mathfrak{f})(p) = (\mathfrak{f}, h(p))_{\mathcal{H}}. \tag{2.1}$$

We suppose that

$$k(p,q) = \langle h(q), h(p)\rangle_{\mathcal{H}}. \tag{2.2}$$

Let $R(\mathcal{L})$ be the range of $\mathcal{L}$ for $\mathcal{H}$ and we introduce the inner product in $R(\mathcal{L})$ induced form the norm

$$\|f\|_{R(\mathcal{L})} = inf\|\mathfrak{f}\|_{\mathcal{H}}, \quad f = \mathcal{L}\mathfrak{f}.$$

Definition 2.2. For the Hilbert space $\mathcal{F}(\lambda)\,(\lambda > 0)$ comprising all entire functions $f(z)$ with finite norms

$$\left\{\frac{\lambda}{\pi}\iint_{\mathbb{C}} |f(z)|^2 e^{-\lambda|z|^2} dxdy\right\}^{\frac{1}{2}} < \infty,$$

the function $e^{\lambda\bar{u}z}$ is the reproducing kernel. This space will be called the Fock space (or the Bargmann–Fock space; see Ref. [14]).

Lemma 2.1. *For a RKHS H_K on E and for any non-vanishing complex-valued function $s(p)$ on E,*

$$k_s(p,q) = s(p)\overline{s(q)}k(p,q), \quad p,q \in E$$

is a reproducing kernel for the Hilbert space H_{k_s} , comprising all the functions $f_s(p)$ on E that are expressible in the form

$$f_s(p) = f(p)s(p), \quad f \in H_K$$

and that is equipped with the inner product

$$\langle f_s, g_s\rangle_{H_{k_s}} = \left\langle \frac{f_s}{s}, \frac{g_s}{s}\right\rangle_{H_K},$$

(see Ref. [12]).

Definition 2.3. Let $\psi(t) \in L^2(R)$, and its Fourier transform $\hat{\psi}(\omega)$ satisfies the admissible condition

$$C_\psi = \int_{\mathbb{R}} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty,$$

then $\psi(t)$ is called the basic wavelet or mother wavelet, and it is also known as an admissible wavelet.