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An iteration method for stable analytic continuation ☆

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ABSTRACT

In the present paper we consider the problem of numerical analytic continuation for an analytic function $f(z)=f(x+iy)$ on a strip domain $\Omega_+=\{z=x+iy\in\mathbb{C}|x\in\mathbb{R},0<y<y_0\}$. The key difference with the known methods is that a novel iteration regularization method with *a priori* and *a posteriori* parameter choice rule is presented. The comparison in numerical aspect with other methods is also discussed.

1. Introduction

The problem of numerical analytic continuation is a very interesting and meaningful topic, and it has many practical applications, e.g., in medical imaging [1], the inversion of Laplace transform [6], inverse scattering problems [10] and so on. Take the scattering problem for example, determining the pion-nucleon coupling constant and the cross sections with unstable particles by analytic continuation of the scattering data are usually encountered, and we refer the reader to reference [2] in detail.

In [7,8], we have considered the following numerical analytic continuation problem with different non-iterative regularization methods.

Problem 1.1. Let

$$\Omega_+=\{z=x+iy\in\mathbb{C}|x\in\mathbb{R},0<y<y_0\}$$

be the strip domain in complex plane $\mathbb{C}$, where i is the imaginary unit and y_0 is a positive constant. The function $f(z)=f(x+iy)$ is an analytic function in $\bar{\Omega}_+$. The data is only given approximately on the real axis $y=0$, i.e., $f(z)|_{y=0}=f(x)$ is known approximately, and the noisy data is denoted by $f^\delta(x)$. We will reconstruct the function $f(z)$ on Ω_+ by using the data $f^\delta(x)$.

This problem is seriously ill-posed and therefore it is very important to study different highly efficient algorithm for it. In [9], the authors give a mollification method with Dirichlet kernel to solve another strip domain $\Omega=\{z=x+iy\in\mathbb{C}|x\in\mathbb{R},|y|<\sigma,\ \sigma>0\,\text{is a constant}\}$. But the mollification in [9] can not be directly applied to the domain Ω_+ since the Dirichlet kernel function cannot be obtained on it. In[7,8] the authors give the Fourier and modified Tikhonov methods, respectively. They obtain order optimal error estimate with effective algorithm. However, error estimates for them only choosing the regularization parameter by *a priori* rule. In the present paper a new regularization method of iteration type

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for solving this problem will be given. Although this method has been used by Deng and Liu to deal with the sideways parabolic equation [3,4], however, we give another way to choose the regularization parameter by an *a posteriori* rule. We will use this method solving Problem 1.1, not only give *a priori* and *a posteriori* rule for choosing regularization parameter with strict theory analysis, but also make comparison in numerical aspects with other methods.

The outline of the paper is as follows. In Section 2, a Hölder-type error estimate is obtained for the *a priori* parameter choice rule. The *a posteriori* parameter choice rule is given in Section 3, which also leads to a Hölder-type error estimate. For the convenience of comparison with other methods, the same numerical examples as in [7,8] are considered in Section 4, which demonstrate the effectiveness of the new method.

2. A priori parameter choice

Let $\hat{g}$ denote the Fourier transform of a function g defined by

$$\hat{g}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} g(x)dx. \tag{2.1}$$

Assume that

$$f(\cdot + iy) \in L^2(\mathbb{R}) \quad \text{for} \quad 0 \leqslant y \leqslant y_0, \tag{2.2}$$

where the norm $\|f\|$ in $L^2(\mathbb{R})$ is defined by:

$$\|f\| := \left(\int_{-\infty}^{\infty} |f(x)|^2 dx \right)^{\frac{1}{2}}.$$

It is easy to know by Parseval formula that there holds $\|f\| = \left(\int_{-\infty}^{\infty} |f(\hat{\xi})|^2 d\xi \right)^{\frac{1}{2}}$. From [7,8], we know

$$\widehat{f(\cdot + iy)}(\xi) = e^{-y\xi} \hat{f}(\xi), \tag{2.3}$$

or equivalently,

$$f(x+iy) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi} e^{-y\xi} \hat{f}(\xi) d\xi. \tag{2.4}$$

Assume the exact data $f(x)$ and the measured data $f_\delta(x)$ both belong to $L^2(\mathbb{R})$, the noise level is given by

$$\|f - f^\delta\| \leqslant \delta, \tag{2.5}$$

and there is an *a priori* bound E such that

$$\|f(\cdot + iy_0)\| \leqslant E. \tag{2.6}$$

For simplicity, we decompose $\mathbb{R}$ into the following parts I and W, where

$$I := \{\xi \in \mathbb{R}, \ \xi \leq 0\}, \quad \text{and} \quad W := \{\xi \in \mathbb{R}, \ \xi \geqslant 0\}. \tag{2.7}$$

For every $\hat{g}(\xi) \in L^2(\mathbb{R})$, define

$$\hat{g}^+(\xi) := \begin{cases} \hat{g}(\xi), & \xi \geqslant 0, \\ 0, & \xi < 0, \end{cases}$$

$$\hat{g}^-(\xi) := \begin{cases} 0, & \xi \geqslant 0, \\ \hat{g}(\xi), & \xi < 0, \end{cases}$$

then $\hat{g}(\xi) = \hat{g}^+(\xi) + \hat{g}^-(\xi)$, and $L^2(\mathbb{R}) = L^2(I) \oplus L^2(W)$.

For $\xi \in W$, the problem is well-posed and we can take the regularization approximation solution in the frequency domain as

$$\widehat{f_k^\delta(\cdot + iy)}(\xi) = e^{-y\xi} \hat{f}^\delta(\xi), \quad \xi \in W, \quad k = 1, 2, \ldots \tag{2.8}$$

For $\xi \in I$, we introduce an iteration scheme as the following form:

$$\widehat{f_k^\delta(\cdot + iy)}(\xi) = (1-\lambda)\widehat{f_{k-1}^\delta(\cdot + iy)}(\xi) + \lambda e^{-y\xi} \hat{f}^\delta(\xi), \quad \xi \in I, \quad k = 1, 2, \ldots \tag{2.9}$$

with initial guess $\widehat{f_0^\delta(\cdot + iy)}(\xi)$, and $\lambda = e^{y_0\xi} < 1$ which plays an important role in the convergence proof. From formula (2.9), it is easy to know

$$\widehat{f_k^\delta(\cdot + iy)}(\xi) = (1-\lambda)^k \widehat{f_0^\delta(\cdot + iy)}(\xi) + \sum_{i=0}^{k-1} (1-\lambda)^i \lambda e^{-y\xi} \hat{f}^\delta(\xi) = (1-\lambda)^k \widehat{f_0^\delta(\cdot + iy)}(\xi) + (1-(1-\lambda)^k) e^{-y\xi} \hat{f}^\delta(\xi). \tag{2.10}$$