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Numerical methods for nonlinear stochastic delay differential equations with jumps

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ABSTRACT

In this paper, we modified the split-step backward Euler method (MSSBE) for stochastic delay differential equations with Poisson-driven jumps (SDDEwJs). Second, we prove that MSSBE is strongly convergent if the drift coefficient $f(x,y)$ satisfies one-side Lipschitz with respect to x, global Lipschitz with respect to y, the diffusion and jump coefficients are globally Lipschitz. On the way to proving the convergence result, we show that Euler–Maruyama method converges strongly when SDDEwJs coefficients satisfy local Lipschitz condition, the pth moments of the exact and numerical solution are bounded for some $p > 2$; the MSSBE may be viewed as an Euler–Maruyama approximation to a perturbed SDDEwJs of the same form.

1. Introduction

Applications in economics, finance, and several areas of science and engineering, give rise to jump-diffusion Ito stochastic differential equations [1,3,12]. [5] have studied the convergence and stability of split-step backward Euler (SBBE) method and compensated SBBE method when the drift coefficients satisfy one-side Lipschitz and the diffusion and jump coefficients are globally Lipschitz. [6] have studied the convergence and stability of implicit methods under global Lipschitz conditions. [7] have shown that under one-side Lipschitz and polynomial growth conditions on the drift coefficient, global Lipschitz conditions on the diffusion and jump coefficients the convergence order of SBBE is 0.5 in mean-square sense. [8] have studied the asymptotic stability of balanced methods for linear stochastic jump-diffusion differential equations.

In general, the future state of a system depends on the present and past states. Hence, it is more significant to consider stochastic delay differential equations with Poisson-driven jumps (SDDEwJs).

Throughout this paper, let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geqslant 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions. Let $W(t)$ be a d-dimensional Brownian motion, $N(t)$ be a scalar Poisson process with intensity λ and independent of the Brownian motion. We will use $|\cdot|$ to denote the Euclidean norm of a vector and the trace norm of a matrix, $\langle \cdot, \cdot \rangle$ to denote the scalar product. We will denote the indicator function of a set G by I_G. For $\mu \in \mathbb{R}$, $\mathrm{In}[\mu]$ denotes the integer part of μ.

Let τ and T be positive constants. In this paper, we consider the following n-dimensional SDDEwJs

$$\begin{cases} dx(t) = f(x(t), x(\tau(t)))dt + g(x(t), x(\tau(t)))dW(t) + h(x(t), x(\tau(t)))dN(t), & 0 \leqslant t \leqslant T, \\ x(t) = \varphi(t), & t \in [-\tau, 0], \end{cases} \tag{1.1}$$

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where $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times d}$, $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $\tau(t)$ satisfy: there exists a positive constant ρ such that

$$-\tau \leqslant \tau(t) \leqslant t, \quad \text{and} \quad |\tau(t) - \tau(s)| \leqslant \rho|t-s|, \quad \forall t,\ s \geqslant 0 \tag{1.2}$$

and $\varphi(t) \in C([-\tau, 0]; \mathbb{R}^n)$ which satisfy: there exist constants $K_1 > 0$ and $\gamma \in (0, 1]$ such that for all $-\tau \leqslant s < t \leq 0$

$$\mathbb{E}|\varphi(t) - \varphi(s)|^2 \leqslant K_1(t-s)^{\gamma}. \tag{1.3}$$

In general, finite-time convergence theory for numerical methods applied to SDDEwJs requires a global Lipschitz assumption on drift, the diffusion and jump coefficients, such as [9,13]. In practice, many important SDDEwJs models satisfy only a local Lipschitz property. We prove that the Euler–Maruyama method converges strongly if f, g and h are locally Lipschiz, the exact and numerical solution have bounded pth moment for some $p > 2$ in Section 2. The bounded moment assumption will not, of course, hold, in general, as solutions to the SDDEwJs may explode in a finite time. In Section 3, we give further assumptions on f, g and h to ensure that $x(t)$ has bounded moments: we assume that g and h are globally Lipschitz, $f(x, y)$ satisfies one-sided Lipschitz condition with respect to x and global Lipschitz condition with respect to y. For a suitably constructed MSSBE, we establish strong convergence (Theorem 3.3) by (a) showing that the method corresponds to Euler–Maruyama on a perturbed SDDEwPJs and (b) showing that all moments of the numerical solution are bounded. We are unable to establish moment bounds for the Euler–Maruyama method and, indeed, it may not be possible to do so [4].

2. The Euler–Maruyama method for locally Lipschitz coefficients

Let the step-size $\Delta \in (0, 1)$ be $\frac{\tau}{m}$ for some positive integer m. The Euler–Maruyama (EM) method applied to (1.1) computers approximations $Y_k \simeq x(t_k)$ with $t_k = k\Delta$, by setting $Y_k = \varphi(t_k)$ for $-m \leqslant k \leq 0$ and forming

$$Y_{k+1} = Y_k + \Delta f(Y_k, Y_{\mathrm{In}[\tau(t_k)/\Delta]}) + g(Y_k, Y_{\mathrm{In}[\tau(t_k)/\Delta]})\Delta W_k + h(Y_k, Y_{\mathrm{In}[\tau(t_k)/\Delta]})\Delta N_k, \tag{2.1}$$

where $\Delta W_k = W(t_{k+1}) - W(t_k)$, $\Delta N_k = N(t_{k+1}) - N(t_k)$.

A key component in our analysis is the compensated Poisson process

$$\widetilde{N}(t) := N(t) - \lambda t,$$

which is a martingale. It is convenient to use continuous-time approximations, and hence we define $\overline{Y}(t)$ by

$$\overline{Y}(t) := Y_0 + \int_0^t f(Y(s), z(s))ds + \int_0^t g(Y(s), z(s))dW(s) + \int_0^t h(Y(s), z(s))dN(s), \tag{2.2}$$

with $\overline{Y}(t) = \varphi(t)$ on $-\tau \leqslant t \leq 0$. Where

$$Y(s) := Y_k, \quad \text{and} \quad z(s) := Y_{\mathrm{In}[\tau(t_k)/\Delta]} \quad \text{for } s \in [t_k, t_{k+1}). \tag{2.3}$$

Our first result makes the following assumption on (1.1), the exact and numerical solutions.

Assumption 2.1. For each $R > 0$ there exists a constant C_R depending on R, such that for all $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$, $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \leqslant R$

$$|a(x, y) - a(\bar{x}, \bar{y})|^2 \leqslant C_R(|x - \bar{x}|^2 + |y - \bar{y}|^2), \quad \text{for } a = f, g \text{ and } h \tag{2.4}$$

For some $p > 2$ there is a constant A such that

$$\mathbb{E}\left[\sup_{-\tau\leqslant t\leqslant T} |x(t)|^p\right] \vee \mathbb{E}\left[\sup_{-\tau\leqslant t\leqslant T} |\overline{Y}(t)|^p\right] \leqslant A. \tag{2.5}$$

We note for later use that linear growth bounds follow straightforwardly:

$$|a(x, y)| \leqslant C_1^R(1 + |x|^2 + |y|^2), \quad \text{for } |x| \vee |y| \leqslant R, \quad a = f, g, h, \tag{2.6}$$

where $C_1^R = 2C_R \vee 2|f(0, 0)|^2 \vee 2|g(0, 0)|^2 \vee 2|h(0, 0)|^2$.

The following result generalises Theorem 2.1 in [11] to the case of jumps.

Theorem 2.2. *Under Assumption 2.1, the EM solution (2.1) with continuous extension (2.2) satisfies*

$$\lim_{\Delta\to 0} \mathbb{E}\left[\sup_{0\leqslant t\leqslant T} |x(t) - \overline{Y}(t)|^2\right] = 0. \tag{2.7}$$

The proof of this theorem is complicated. Next, we will present two lemmas. First, we define

$$\rho_R := \inf\{t \geqslant 0 : |x(t)| \geqslant R\}, \quad \tau_R := \inf\{t \geqslant 0 : |\overline{Y}(t)| \geqslant R\}, \theta_R := \rho_R \wedge \tau_R.$$