Applied Mathematics and Computation 233 (2014) 310–319

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

# On fourteen solvable systems of difference equations

D.T. Tollu [a], Y. Yazlik [b,*], N. Taskara [c]

[a] Department of Mathematics–Computer Sciences, Faculty of Science, Necmettin Erbakan University, 42090 Konya, Turkey
[b] Department of Mathematics, Faculty of Science and Art, Nevsehir Haci Bektas Veli University, 50300 Nevsehir, Turkey
[c] Department of Mathematics, Faculty of Science, Selcuk University, Campus, 42075 Konya, Turkey

ARTICLE INFO

*Keywords:*
System of difference equations
Riccati difference equation
General solution
Fibonacci numbers

ABSTRACT

In this paper, we mainly consider the systems of difference equations

$$x_{n+1} = \frac{1+p_n}{q_n}, \quad y_{n+1} = \frac{1+r_n}{s_n}, \quad n \in \mathbb{N}_0,$$

where each of the sequences $p_n, q_n, r_n$ and $s_n$ represents either the sequence $x_n$ or the sequence $y_n$, with nonzero real initial values $x_0$ and $y_0$. Then we solve fourteen out of sixteen possible systems. It is noteworthy to depict that the solutions are presented in terms of Fibonacci numbers for twelve systems of these fourteen systems.


## 1. Introduction and preliminaries

Studying difference equations and their systems have recently taken much attention (see [1–20] and the references cited in them).

One of the most interesting examples of those equations is called *Riccati difference equation* and defined by

$$x_{n+1} = \frac{a+bx_n}{c+dx_n}, \quad n \in \mathbb{N}_0, \tag{1}$$

with a real initial value $x_0$. In the literature, there are so many studies on Eq. (1) [cf. 1–3, 15–18]. For $a = 0$ and $b = c = d = 1$, we note that Eq. (1) reduces to the $x_{n+1} = \frac{x_n}{1+x_n}, n \in \mathbb{N}_0$, which has actually the general solution

$$x_n = \frac{x_0}{x_0 n + 1}, \quad n \in \mathbb{N}_0. \tag{2}$$

For real initial values $x_0$ and $y_0$, Stević ([3]) showed that the systems

$$x_{n+1} = \frac{u_n}{1+v_n}, \quad y_{n+1} = \frac{w_n}{1+s_n}, \quad n \in \mathbb{N}_0 \tag{3}$$

are solvable in fourteen out of sixteen possible cases such that $u_n, v_n, w_n, s_n$ are some of the sequences $x_n$ or $y_n$ by their own. In that reference, the author used the formula given in (2) as well as several tricks and methods to solve the related systems. Moreover, by keeping this study in the mind, one can claim that there may be other systems of difference equations which can be solved in a similar way. Therefore, we consider the systems of difference equations

* Corresponding author.
*E-mail addresses:* dttollu@konya.edu.tr (D.T. Tollu), yyazlik@nevsehir.edu.tr (Y. Yazlik), ntaskara@selcuk.edu.tr (N. Taskara).

http://dx.doi.org/10.1016/j.amc.2014.02.001
0096-3003/

$$x_{n+1}=\frac{1+p_n}{q_n},\quad y_{n+1}=\frac{1+r_n}{s_n},\quad n\in\mathbb{N}_0, \tag{4}$$

where each of the sequences $p_n, q_n, r_n$ and $s_n$ represents one of the sequences $x_n$ and $y_n$, with nonzero real initial values $x_0$ and $y_0$. Thus, by establishing the existence of sixteen possible systems derived from (4), we will mainly show that fourteen systems are solvable while remaining two are not. At this point, we strictly note that the formulas of some solutions will be presented in terms of the Fibonacci numbers. During to solve the systems mentioned above, we will take into account some methods given in [3,20].

To catch the aim of the paper, let us give the following preliminary information.

If $d\neq 0$, by using the substitution $x_n=\left(\frac{b+c}{d}\right)y_n-\frac{c}{d}$, then Eq. (1) reduces to the one parameter equation

$$y_{n+1}=\frac{-R+y_n}{y_n},\quad n\in\mathbb{N}_0, \tag{5}$$

where $R=\frac{bc-ad}{(b+c)^2}$ is called the *Riccati number*. For $R=-1$, when we use the substitution $y_n=\frac{z_{n+1}}{z_n}$, Eq. (5) becomes

$$z_{n+2}-z_{n+1}-z_n=0,\quad n\in\mathbb{N}_0, \tag{6}$$

where the initial values $z_0, z_1\in\mathbb{R}\setminus\{0\}$. We note that Eq. (5) has the characteristic equation

$$\lambda^2-\lambda-1=0,$$

with characteristic roots $\lambda_+=\frac{1+\sqrt{5}}{2}$ and $\lambda_-=\frac{1-\sqrt{5}}{2}$. On the other hand, it is well-known that Eq. (6) has the solution

$$z_n=\frac{(z_1-\lambda_-z_0)\lambda_+^n-(z_1-\lambda_+z_0)\lambda_-^n}{\lambda_+-\lambda_-},\quad n\in\mathbb{N}_0. \tag{7}$$

It is easy to see that the equality in (7) can be written as the form

$$z_n=F_nz_1+F_{n-1}z_0,\quad n\in\mathbb{N}_0, \tag{8}$$

where $F_n$ is $n$th *Fibonacci number* defined by $F_n=\frac{\lambda_+^n-\lambda_-^n}{\lambda_+-\lambda_-}$. By substituting (8) in $y_n=\frac{z_{n+1}}{z_n}$, we obtain

$$y_n=\frac{F_{n+1}y_0+F_n}{F_ny_0+F_{n-1}} \tag{9}$$

for the general solution of (5) if $R=-1$.

We will also consider the linear equation

$$t_{n+2}+t_{n+1}-t_n=0,\quad n\in\mathbb{N}_0, \tag{10}$$

with real initial values $t_0$ and $t_1$ which will be used in the sequel. Note that the characteristic equation of Eq. (10) is $\mu^2+\mu-1=0$ such that $\mu_\pm=-\lambda_\mp$. In fact, both this last equality and (8) imply that

$$t_n=(-1)^{n-1}(F_nt_1-F_{n-1}t_0),\quad n\in\mathbb{N}_0, \tag{11}$$

where $F_n$ is the $n$th Fibonacci number as defined above.

The following lemma will be needed in our calculations.

**Lemma 1.1** [21]. *Let $F_n$ be nth Fibonacci number. Then the following statements hold:*

(i) $\sum_{i=1}^{n}F_i=F_{n+2}-1$,
(ii) $\lim_{n\to\infty}\frac{F_{n+1}}{F_n}=\frac{1+\sqrt{5}}{2}=\lambda_+$.

## 2. Main results

The main purpose of this section is to solve the above mentioned sixteen systems of difference equations. By assuming the solutions are well-defined, we will give the general solutions of fourteen systems.

### *2.1. Case 1: $p_n=x_n,\ q_n=x_n,\ r_n=y_n,\ s_n=y_n$*

In this case, the system is expressed as

$$x_{n+1}=\frac{1+x_n}{x_n},\quad y_{n+1}=\frac{1+y_n}{y_n},\quad n\in\mathbb{N}_0. \tag{12}$$

From (9), the general solution follows straightforwardly as

$$x_n=\frac{F_{n+1}x_0+F_n}{F_nx_0+F_{n-1}},\quad y_n=\frac{F_{n+1}y_0+F_n}{F_ny_0+F_{n-1}},\quad n\in\mathbb{N}_0. \tag{13}$$