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An explicit iteration for zeros of accretive operators

Yisheng Song [a,b,*], ChangAn Tian [a]

[a] College of Mathematics and Information Science, Henan Normal University, XinXiang, HeNan 453007, PR China
[b] Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong

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ABSTRACT

In this paper, for Lipschitz accretive operator A, an iteration scheme is defined as follows:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n(u - \beta_n Ax_n).$$

Its strong convergence is established for finding some zero of A whenever $\alpha_n, \beta_n \in (0,1)$ satisfying conditions:

$$\lim_{n\to\infty}\alpha_n = 0, \quad \sum_{n=1}^{+\infty}\alpha_n = +\infty, \quad \lim_{n\to\infty}\beta_n = 0.$$

Furthermore, some applications for equilibrium problems are given also. In particular, the iteration coefficient is simpler and more general.

1. Introduction

Throughout this paper, a Banach space E will always be over the real scalar field. We denote its norm by $\|\cdot\|$ and its dual space by E^*. The value of $x^* \in E^*$ at $y \in E$ is denoted by $\langle y, x\rangle$ and the *normalized duality mapping* J from E into 2^{E^*} is defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\|f\|, \|x\| = \|f\|\}, \quad \forall x \in E.$$

It is well known (see, for example, [28, Theorems 4.3.1, 4.3.2]) that E is smooth (equivalently, E^* is strict convex [28, p. 113, Problem 3]) if and only if J is single-valued. In the sequel, we shall denote the single-valued normalized duality map by j. Let $A^{-1}0 = \{x \in D(A); 0 \in Ax\}$, the set of zeros of an operator A and $\mathbb{N}$ denote the set of all positive integer. We write $x_n \rightharpoonup x$ (respectively $x_n \overset{*}{\rightharpoonup} x$) to indicate that the sequence x_n weakly (respectively weak*) converges to x; as usual $x_n \to x$ will symbolize strong convergence.

Recall a mapping A with domain $D(A)$ and range $R(A)$ in E is called *Lipschitzian* if for all $x, y \in D(A)$, there exists $L > 0$ such that

$$\|Ax - Ay\| \leqslant L\|x - y\|.$$

In particular, A is called *nonexpansive* whenever $L = 1$ and *contractive* when $0 \leqslant L < 1$. A mapping $A : D(A) \subset E \to 2^E$ is called to be *accretive* if for all $x, y \in D(A)$ there exists $j(x-y) \in J(x-y)$ such that

$$\langle u - v, j(x-y)\rangle \geqslant 0, \quad \text{for } u \in Ax \text{ and } v \in Ay;$$

* Corresponding author at: College of Mathematics and Information Science, Henan Normal University, XinXiang, HeNan 453007, PR China.
E-mail address: songyisheng1@gmail.com (Y. Song).

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If E is a Hilbert space, accretive operators are also called monotone. An operator A is called $m-$*accretive* if it is accretive and $R(I+rA)$, range of $(I+rA)$, is E for all $r>0$; A is said to *satisfy the range condition* if $\overline{D(A)} \subset R(I+rA), \forall r>0$, where I is the identity operator of E and $\overline{D(A)}$ denotes the closure of the domain of A.

Interest in accretive operators stems mainly from their firm connection with equations of evolution. It is well known (see [33]) that many physically significant problems can be modeled by the initial-value problems of the form

$$\begin{cases} x'(t) + Ax(t) = 0, \\ x(0) = x_0, \end{cases} \quad (1.1)$$

where A is an accretive operator in an appropriate Banach space. Typical examples where such evolution equations occur can be found in the heat and wave equations or Schrodinger equations. Especially, one of the fundamental results in the theory of accretive operators, which is due to Browder [3], states that, if A is locally Lipschitzian and accretive, then A is m-accretive. This result was subsequently generalized by Martin [14] to the continuous accretive operators. If, in (1.1), $x(t)$ is independent of t, then (1.1) reduces to $Au=0$ whose solutions corresponds to the equilibrium points of the system (1.1). Consequently, considerable research effects have been devoted, especially, within the past 20 years or so, to the iterative methods for approximating these equilibrium points.

One popular method of solving $0 \in Ax$ is *Rockafellar proximal point algorithm* [20] which is recognized as a powerful and successful algorithm in finding a zero of accretive operators. Starting from any initial guess $x_0 \in E$, this proximal point algorithm generates a sequence $\{x_k\}$ according to the inclusion:

$$x_k + e_k \in x_{k+1} + c_k A(x_{k+1}),$$

where $\{e_k\}$ is a sequence of errors and $\{c_k\}$ is a sequence of positive regularization parameters. Note that the above algorithm can be rewritten as

$$x_{k+1} = J^A_{c_k}(x_k + e_k), \quad (1.2)$$

where $J^A_r = (I+rA)^{-1}$ for all $r>0$, is the resolvent of A with I being the identity map on the space E. Rockafellar [20] proved the weak convergence of his algorithm (1.2) provided the regularization sequence $\{c_k\}$ remains bounded away from zero and the error sequence $\{e_k\}$ satisfies the condition $\sum_{k=0}^{+\infty}\|e_k\| < \infty$. So to have strong convergence, one has to modify the algorithm (1.2). Several authors proposed modifications of Rochafellar's proximal point algorithm (1.2) to have strong convergence. Solodov–Svaiter [27] initiated such investigation followed by Kamimura–Takahashi [11] (in which the work of [11] is extended to the framework of uniformly convex and uniformly smooth Banach spaces). Bruck [1] introduced an iteration process and proved, in Hilbert space setting, the convergence of the process to a zero of a maximal monotone operator. In 1979, Reich [18] extended this result to uniformly smooth Banach spaces provided that the operator is m-accretive. Reich [19] unified the above results and proved the following (also see Takahashi and Ueda [29 Theorem 1]):

Theorem R [19, Theorem 1, Remark]. *Let E be a reflexive Banach space whose norm is uniformly Gâteaux differentiable. Suppose that every weakly compact convex subset of E has the fixed point property for non-expansive mappings. Assumed that $A: D(A) \subset E \to 2^E$ be an accretive operator with resolvent J^A_r for $r>0$ and $A^{-1}0 \neq \emptyset$, and K is a nonempty closed convex subset of E such that $\overline{D(A)} \subset K \subset \bigcap_{r>0} R(I+rA)$. If $0 \in R(A)$, then the strong $\lim_{r\to\infty} J^A_r u$ exists and belongs to $A^{-1}0$ for each $u \in K$. Further, if $Pu = \lim_{r\to\infty} J^A_r u$ for each $u \in K$, then P is the unique sunny nonexpansive retraction of K onto $A^{-1}0$.*

By the inspiration for the Rockafellar proximal point algorithm and the iterative methods of Halpern [10], Dominguez Benavides et al. [4] studied the Halpern type iteration process (1.2) to find a zero of an $m-$accretive operator A in a uniformly smooth Banach space with a weakly continuous duality mapping J_φ with gauge φ in virtue of the resolvent J^A_r of A:

$$x_{n+1} = \alpha_n u + (1-\alpha_n) J^A_{r_n} x_n, \quad (1.3)$$

where $\{\alpha_n\}$ and $\{\beta_n\} \subset (0,1)$ and $\{r_n\} \subset (0,+\infty)$ satisfying the conditions:

$$\lim_{n\to\infty}\alpha_n = 0,\ \sum_{n=1}^{+\infty}\alpha_n = +\infty,\ \lim_{n\to\infty} r_n = +\infty.$$

Xu [32] and Marino and Xu [13] also researched the above iteration process in a uniformly smooth Banach space.

Recently, Xu [30] and Song and Yang [26] studied the strong convergence of the regularization method for Rockafellar's proximal point algorithm of the resolvent J_r in a Hilbert space:

$$x_{n+1} = J^A_{r_n}(\alpha_n u + (1-\alpha_n)x_n), \quad (1.4)$$

where $\{\alpha_n\} \subset (0,1)$ and $\{r_n\} \subset (0,+\infty)$ satisfying the conditions:

$$\lim_{n\to\infty}\alpha_n = 0,\ \sum_{n=1}^{+\infty}\alpha_n = +\infty,\ \sum_{n=1}^{+\infty}|\alpha_{n+1}-\alpha_n| < +\infty,\ \sum_{n=1}^{+\infty}|r_{n+1}-r_n| < +\infty.$$

Song [21] established the strong convergence for two explicit iteration schemes for approaching a zero of an accretive operator A:

$$x_{n+1} = \alpha_n u + \beta_n x_n + (1-\alpha_n-\beta_n) J^A_{r_n} x_n, \quad (1.5)$$

$$x_{n+1} = \beta_n x_n + (1-\beta_n) J^A_{r_n}(\alpha_n u + (1-\alpha_n)x_n), \quad (1.6)$$