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Global asymptotic stability of a second order rational difference equation

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ABSTRACT

The main goal of this paper is to investigate the global asymptotic stability of the difference equation

$$y_{n+1} = \frac{p_n + y_n}{p_n + y_{n-1}}, \quad n = 0, 1, 2, \ldots,$$

where

$$p_n = \begin{cases} \alpha, & \text{if } n \text{ is even} \\ \beta, & \text{if } n \text{ is odd} \end{cases} \quad \text{and} \quad \alpha > 0,\ \beta > 0,\ \alpha \neq \beta$$

and the initial conditions $y_{-1}, y_0 \in [0, \infty)$. We show that the unique equilibrium $\bar{y} = 1$ is globally asymptotically stable under the certain conditions.

1. Introduction and preliminaries

Our aim in this paper is to investigate the global stability of the following difference equation

$$y_{n+1} = \frac{p_n + y_n}{p_n + y_{n-1}}, \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

where

$$p_n = \begin{cases} \alpha, & \text{if } n \text{ is even} \\ \beta, & \text{if } n \text{ is odd} \end{cases} \quad \text{and} \quad \alpha > 0,\ \beta > 0,\ \alpha \neq \beta$$

and the initial conditions $y_0, y_{-1} \in [0, \infty)$. In [10, Open Problem 4.8.12], Kulenović and Ladas proposed the following open problem. See also [3, Open Problem 5.26.1].

Open Problem 1.1. Let $\{p_n\}_{n=0}^{\infty}$ be a periodic-two sequence of nonnegative real numbers. Investigate the global character of all positive solutions of Eq. (1.1).

Recently, systematic analysis of difference equations with periodic coefficients was considered by several authors, see, for example Refs. [6,7,11]. Inspired by the above open problem, we investigate the global stability of Eq. (1.1) with a periodic-two coefficients, which is the non-autonomous equation that corresponds to the following autonomous equation

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$$y_{n+1} = \frac{p + y_n}{p + y_{n-1}}, \quad n = 0, 1, 2, \ldots, \quad p > 0, \; y_{-1}, \; y_0 \in [0, \infty). \tag{1.2}$$

In [9], it is shown that the unique equilibrium $\bar{y} = 1$ of Eq. (1.2) is globally asymptotically stable. See also [8, p.73] and [3, p.168]. Our main goal in this paper is to extend the result to the non-autonomous difference Eq. (1.1) under the certain conditions.

Before our discussion, we present some definitions and the known results which will be useful in the sequel. For the general theory of difference equations, one can refer to the monographes of Kocic and Ladas [8] and Kulenović and Ladas [10]. For other related results on nonlinear difference equations, see, for example, [1–16] and the references therein.

Consider the system:

$$\begin{cases} u_{n+1} = f(u_n, v_n), \\ v_{n+1} = g(u_n, v_n). \end{cases} \quad n = 0, 1, \ldots. \tag{1.3}$$

Let $\| \cdot \|$ be the norm of vector $(u, v) \in \mathbb{R}^2$. Then, we present some definitions and some useful lemmas.

Definition 1.2. The equilibrium point $(\bar{u}, \bar{v})$ is said to be:

(i) stable if given $\epsilon > 0$ and $N > 0$ there exists $\delta > 0$ such that $\|(u_0, v_0) - (\bar{u}, \bar{v})\| < \delta$ implies that $\|(u_n, v_n) - (\bar{u}, \bar{v})\| < \epsilon$ for all $n > N$, and unstable if it is not stable;
(ii) attracting if there exists $\eta > 0$ such that $\|(u_0, v_0) - (\bar{u}, \bar{v})\| < \eta$ implies that $\lim_{n \to \infty}(u_n, v_n) = (\bar{u}, \bar{v})$;
(iii) asymptotically stable if it is stable and attracting.

The main result in linearized stability analysis is following lemma which is extracted from [4], see also [10].

Lemma 1.3. *Let $F = (f, g)$ be a continuously differentiable function defined on an open set $D \subset \mathbb{R}^2$.*

(a) *If the eigenvalues of the Jacobian matrix $J_F((\bar{u}, \bar{v}))$, that is, both roots of its characteristic equation*

$$\lambda^2 - TrJ_F((\bar{u}, \bar{v}))\lambda + DetJ_F((\bar{u}, \bar{v})) = 0, \tag{1.4}$$

lie inside the unit disk, then the equilibrium $(\bar{u}, \bar{v})$ of Eq. (1.3) is locally asymptotically stable.
(b) *A necessary and sufficient condition for both roots of Eq. (1.4) to lie inside the unit disk is*

$$|TrJ_F((\bar{u}, \bar{v}))| < 1 + DetJ_F((\bar{u}, \bar{v})) < 2.$$

In this case, the locally asymptotically stable equilibrium $(\bar{u}, \bar{v})$ is also called a sink.

The following Lemma can be founded in [12].

Lemma 1.4. *Let $B = [a, b] \times [c, d] \in \mathbb{R}^2$. Assume that $F = (f, g) : B \to B$ is a continuous function satisfying the following properties:*

(i) *$f(u, v)$ is non-increasing in the first variable and non-decreasing in the second variable and g is non-increasing in each of its variables for each $(u, v) \in B$;*
(ii) *If $(m, M; l, L) \in B \times B$ is a solution of the system*

$$\begin{cases} M = f(m, L), & m = f(M, l), \\ L = g(m, l), & l = g(M, L). \end{cases}$$

then $m = M$ and $l = L$.

Then the system (1.3) has a unique equilibrium $(\bar{u}, \bar{v})$ and every solution of Eq. (1.3) with $(u_0, v_0) \in B$ converges to the equilibrium $(\bar{u}, \bar{v})$.

2. Period-two character and linearized stability

Theorem 2.1. *Eq. (1.1) has no positive prime period-two solution.*

Proof. Assume for the sake of contradiction that

$$\ldots, \phi, \varphi, \phi, \varphi, \ldots$$

is a positive prime period-two solution of Eq. (1.1), then