Applied Mathematics and Computation 233 (2014) 463–479

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

The hexanomial lattice for pricing multi-asset options

Wen-Hung Kao [a], Yuh-Dauh Lyuu [b,c,*], Kuo-Wei Wen [b]

[a] Department of Derivatives, Capital Securities Corp. Capital Center, No. 101, Songren Rd., Taipei 11073, Taiwan
[b] Department of Computer Science and Information Engineering, National Taiwan University, No. 1, Sec. 4, Roosevelt Rd., Taipei 10617, Taiwan
[c] Department of Finance, National Taiwan University, No. 1, Sec. 4, Roosevelt Rd., Taipei 10617, Taiwan

ARTICLE INFO

Keywords:
Binomial lattice
Trinomial lattice
Hexanomial lattice
Multi-asset option
Barrier option
Correlation

ABSTRACT

Multi-asset options are important financial derivatives. Because closed-form solutions do not exist for most of them, numerical alternatives such as lattice are mandatory. But lattices that require the correlation between assets to be confined to a narrow range will have limited uses. Let ρ_{ij} denote the correlation between assets i and j. This paper defines a (correlation) optimal lattice as one that guarantees validity as long as $-1 + O(\sqrt{\Delta t}) \leqslant \rho_{ij} \leqslant 1 - O(\sqrt{\Delta t})$ for all pairs of assets i and j, where Δt is the duration of a time period. This paper then proposes the first optimal bivariate lattice (generalizable to higher dimensions), called the hexanomial lattice. This lattice furthermore has the flexibility to handle a barrier on each asset. Experiments confirm its excellent numerical performance compared with alternative lattices.

1. Introduction

Options are financial derivatives whose payoff depends on some underlying assets [18]. The underlying assets can be stocks, bonds, currencies, interest rates, volatilities, commodities, temperature, and so on. A fundamental result in finance says the theoretical price of an option equals the discounted expected payoff under the risk-neutral probability measure [28].

Multi-asset options have a payoff determined by multiple underlying assets, and they are essential tools for speculation and risk management. They are also called rainbow options and correlation options [28,33]. These options are challenging to price because of problems arising from their high dimensions and the correlations between assets. Table 1 samples a few popular multi-asset options with their payoff functions specified.

Multi-asset options can be priced in many ways. Because of the curse of dimensionality, Monte Carlo simulation and its variants are the most popular general-purpose approaches [5]. Joy et al. [21], Boyle et al. [9], and Tan and Boyle [37] apply quasi Monte Carlo simulation to price multi-asset European and American options. Quasi Monte Carlo methods in general have faster convergence rates and generate better confidence intervals than the standard Monte Carlo method [40].

Simulation is relatively time consuming and does not converge fast, however. Closed-form pricing formulas, in contrast, do not suffer from the same problems if the dimension of the resulting integral is reasonably small. Stulz presents closed-form formulas for bivariate maximum and minimum options [36]. Johnson extends them to multi-asset maximum and

* Corresponding author at: Department of Computer Science and Information Engineering, Department of Finance, National Taiwan University, No. 1, Sec. 4, Roosevelt Rd., Taipei 10617, Taiwan.
E-mail addresses: vanhongkao@gmail.com (W.-H. Kao), lyuu@csie.ntu.edu.tw (Y.-D. Lyuu), wenkuowei@gmail.com (K.-W. Wen).

http://dx.doi.org/10.1016/j.amc.2014.01.173
0096-3003/

Table 1
A sampling of multi-asset options.

Type	Payoff
Exchange option [29]	$\max(S_1(T) - S_2(T), 0)$
Better-off option [41]	$\max(S_1(T), \ldots, S_n(T))$
Worse-off option [41]	$\min(S_1(T), \ldots, S_n(T))$
Binary maximum option	$\mathbf{1}_{\{\max(S_1(T),\ldots,S_n(T))>K\}}$
Maximum option [20,36]	$\max(\max(S_1(T), \ldots, S_n(T)) - K, 0)$
Minimum option [20,36]	$\max(\min(S_1(T), \ldots, S_n(T)) - K, 0))$
Spread option [12,30]	$\max\,(S_1(T) - S_2(T) - K, 0)$
Basket average option [20,23]	$\max((S_1(T) + \cdots + S_n(T))/n - K, 0)$
Multi-strike option [41]	$\max(S_1(T) - K_1, \ldots, S_n(T) - K_n, 0)$
Pyramid rainbow option [41]	$\max(\lvert S_1(T) - K_1\rvert + \cdots + \lvert S_n(T) - K_n\rvert - K, 0)$
Madonna rainbow option [41]	$\max\left(\sqrt{(S_1(T)-K_1)^2+\cdots+(S_n(T)-K_n)^2}-K,0\right)1$

$S_i(T)$ denotes the price of the ith underlying asset at the maturity date T, K_i is the strike price for the ith underlying asset, and **1** is the indicator function.

minimum options [20]. Kirk and Aron's [24] approximation method is able to price bivariate spread options, but it is inaccurate when the strike prices are high [2,12,30].

Although closed-form solutions are often preferred, they are rare. To fill this void, partial differential equations (PDEs) and the closely related lattices are popular numerical alternatives [1,6,7,27,38]. Both methods divide the time into discrete time periods with a duration of Δt. There are two general types of finite-difference methods: the explicit and implicit ones The explicit methods of Brennan and Schwartz [4], Boyle and Tian [10], and Hull and White [17] can be used to price bivariate options and they are conditionally stable. (A bivariate option is a multi-asset option with two underlying assets.) Unconditionally stable implicit methods such as Kurpiel and Roncalli's [25] and Gillia et al.'s [16] are also able to price bivariate options. Although the explicit method is conditionally stable, it is both easier to implement and conceptually simpler than the implicit method. For example, the explicit method for a variety of derivative security pricing problems uses between 40% and 70% as much time as the implicit method to provide the same level of accuracy [17]. Therefore, there does not seem to be a clear winner between the implicit method and the explicit one.

A lattice is essentially an explicit finite-difference method to solve the pricing PDE [28]. A lattice for multi-asset options is called a multi-asset lattice; in particular, a lattice for bivariate options is called a bivariate lattice. With few exceptions, multi-asset lattices are either binomial or trinomial. In the former case, each asset's price may move up or down (hence 2 branches). In the latter case, each asset's price move has 3 choices (hence 3 branches). The number of jumps of a multi-asset lattice is the total number of branches emanating from each node. Many multi-asset lattices have been proposed. The bivariate lattice of Boyle et al. has 4 jumps as each asset's price moves follow the binomial branch [7]. The same is true of the bivariate lattice of Rubinstein, who calls it a pyramid [34]. The bivariate lattice of Boyle has 5 jumps as each asset's price moves follow the trinomial branch [6]. A high-dimensional extension of the above-mentioned lattice of Boyle [6] is the trinomial lattice of Kamrad and Ritchken. It contains $2^k + 1$ jumps when there are k underlying assets [22].

A lattice should not have invalid transition probabilities [42]. But some multi-asset lattices have invalid transition probabilities unless the correlation between assets falls within a typically narrow range. Let ρ_{ij} denote the correlation between assets i and j. (For brevity, the correlation is simply ρ for the bivariate case.) To highlight the key role played by ρ_{ij} in the construction of valid lattices, this paper calls a lattice (correlation) optimal if the transition probabilities are guaranteed to be valid as long as $-1 + O(\sqrt{\Delta t}) \leqslant \rho_{ij} \leqslant 1 - O(\sqrt{\Delta t})$ for all pairs of assets i and j. Note that optimality here concerns correlation not running time. An optimal lattice, therefore, is guaranteed to be valid for essentially all possible correlations. Optimality is clearly a desirable property for lattices. But, surprisingly, this important feature has been overlooked in the literature.

This paper analyzes existing multi-asset lattices for their optimality. The 5-jump bivariate lattice of Boyle [6], the 4-jump bivariate lattice of Boyle et al. [7], and the 4-jump bivariate lattice of Rubinstein [34] are shown to be optimal. Lin's 9-jump bivariate lattice is however suboptimal [26].

This paper proceeds to propose a new lattice, called the hexanomial lattice, that is provably optimal in the bivariate case. In the construction, a time period is divided into two phases. A 4-jump binomial structure is employed in the first phase and a 9-jump trinomial structure in the second. The result is a 36-jump lattice, with 6 branches in each asset. An additional advantage of the hexanomial lattice is its ability to price options with multiple barriers (to be elaborated shortly).

Orthogonalization can also be used to construct optimal lattices [11,17,39,41]. It transforms the processes for the assets' prices into uncorrelated ones. A valid lattice is then built for each process independently. They are later combined to form a multi-asset lattice. Finally, the uncorrelated processes are transformed back to the original assets' prices. In contrast, the hexanomial lattice works directly with assets' prices instead of variables which are linear functions of assets' prices. The connection to the payoff function is thus transparent, whereas orthogonalization obscures it somewhat.

A common challenge facing a lattice is pricing barrier options with good convergence behavior. The payoff of a barrier option depends on whether an asset's price ever touches a certain price level called the barrier. There are two general types of barrier options. A knock-out option immediately terminates if an asset's price touches the barrier. In contrast, a knock-in