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Lagrangian description of gravity-capillary waves propagating on a sloping bottom

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ABSTRACT

An asymptotic solution that describes a small amplitude gravity-capillary wave propagating on the surface of a gentle sloping beach is derived in the Lagrangian coordinates. The analytical solution in Lagrangian form satisfies the zero pressure at the free surface. In the Lagrangian approximation, the parametric expression of water particles can be obtained directly and explicitly as a function of the wave steepness, the bottom slope and surface tension. The analytical solution for wave asymmetry parameter up to the breaker line for an arbitrary bottom slope can also be derived. The Lagrangian solution enables the description of the features of wave shoaling in the direction of wave propagation from deep to shallow water, as well as the process of successive deformation of a wave profile which leads to wave breaking. Furthermore, by comparing the theoretical values of wave asymmetry with experimental results, it is found that theoretical results of the present solution are in good agreement with the experimental data. It is also found that surface tension lower the breaking wave height, lengthen the wave length and increase the breaking water depth.

1. Introduction

The motion of fluid flow may be described by either observing the fluid velocity at a fixed position or the trajectory of a particle that is carried along with the flow. These alternative descriptions are called Eulerian and Lagrangian method, respectively. For an incompressible fluid the Eulerian approach is clearly preferable, because of the corresponding continuity equation is linear. It is known that the Eulerian description at the free surface is always expressed in Taylor series at a fixed water level, which implicitly assumes that the surface profile is a differentiable single-valued function. Unlike an Eulerian surface, which is given as an implicit function, a Lagrangian form is expressed through a parametric representation of particle motion. A Lagrangian description is more appropriate for limiting free surface motion whereas this feature cannot be represented by the classical Eulerian solutions [20–22].

The first water wave theory in Lagrangian coordinates was obtained by Gerstner [18] who assumed the flow possesses finite vorticity. After Gerstner's original discovery, this wave motion was re-discovered by Rankine [29].Gerstner's wave is a periodic travelling wave with a specific vorticity distribution (see [7,19] for a modern treatment of Gerstner's wave). Miche [26] proposed perturbation Lagrangian solution to the second order for a gravity wave. Pierson [28] also applied perturbation

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expansion to water wave problems with Lagrangian formulae and obtained a first-order Lagrangian solution. Buldakov et al. [3] developed a Lagrangian asymptotic formulation up to a fifth order for nonlinear water waves in the deep water. However, the theories mentioned above are highly rotational, which is inconsistent with Kelvin's circulation theorem [25,31] and are limited to the condition of uniform water depth.

To date, only a limited few analytic solutions are derived for wave transformation on a planar beach in Lagrangian coordinates. Among them, Sanderson [30] obtained a second-order solution in a uniformly stratified fluid with a small bottom slope in a Lagrangian system. Constantin [6] considered the first-order Lagrangian solution for edge wave on a sloping beach in a homogeneous flow. This solution was extended to stratified flows [32]. Kapinski [24] studied the runup of a long wave over a uniform sloping bottom in Lagrangian description. Chen and Hwung[5] obtained linear solution in a uniformly fluid with small bottom slopes in the Lagrangian system. The particle path description was recently addressed in the context of the 2004 tsunami [13]; see also the related discussion of Constantin [9]. However, Chen and Huang[5] did not consider the effect of surface tension. Its well known that surface tension generates a significant influence on the wave breaking profile, limiting wave height and the breaker height [1,4,27].

The purpose of this paper is devoted to explicit Lagrangian asymptotic solution for gravity-capillary waves propagating over a sloping bottom in Lagrangian coordinates. In order to examine the effects of sloping bottoms and surface tension on surface waves, a perturbation expansion is used to derive an expression of the particle trajectories in terms of the wave steepness and the bottom slope. The free surface conditions in Lagrangian coordinates are linear and the physical quantities related to the wave motion are expanded in terms of the bottom slope, wave steepness and surface tension, so that the asymptotic solution in Lagrangian coordinates could be derived. The wave profile is obtained by setting vertical label equal to zero at free surface in which the zero pressure is satisfied. Finally, discussions are drawn for the related physical quantities obtained from the present solutions.

2. Formulation of the problem

Consider a two-dimensional monochromatic wave propagating normally over a uniform gentle slope as shown in Fig. 1. The negative x-axis directed outward to the sea, while the positive y-axis taken vertically upward from the still water level, where the sea bottom is at $y = -d = \alpha x$, in which denotes the bottom slope. The fluid motion in the Lagrangian representation is described by tracing individual fluid particles. For two-dimensional flow, fluid particles are distinguished by the horizontal and vertical parameters $x_0,\ y_0$, known as Lagrangian labels. These labels have one-to-one correlation with the initial positions (x, y) of particles which has been shown in, say, Section 16 of Lamb [25] or Yakubovich and Zenkovich [34]. The fluid motion is described by a set of trajectories $x(x_0, y_0, t)$ and $y(x_0, y_0, t)$, where x and y are Cartesian coordinates. The dependent variables x and y express the position of any particle at time t and are function of the independent variables $x_0,\ y_0$ and t. The system of Lagrangian governing equations and boundary conditions for two-dimensional irrotational free-surface flow are summarized below:

$$x_{x_0}y_{y_0} - x_{y_0}y_{x_0} = \frac{\partial(x,y)}{\partial(x_0,y_0)} = 1, \tag{1}$$

$$x_{x_0t}y_{y_0} - x_{y_0t}y_{x_0} + x_{x_0}y_{y_0t} - x_{y_0}y_{x_0t} = \frac{\partial(x_t,y)}{\partial(x_0,y_0)} + \frac{\partial(x,y_t)}{\partial(x_0,y_0)} = 0, \tag{2}$$

$$x_{x_0t}x_{y_0} - x_{y_0t}x_{x_0} + y_{x_0t}y_{y_0} - y_{y_0t}y_{x_0} = \frac{\partial(x_t,x)}{\partial(x_0,y_0)} + \frac{\partial(y_t,y)}{\partial(x_0,y_0)} = 0, \tag{3}$$

$$\frac{\partial\phi}{\partial x_0} = x_t x_{x_0} + y_t y_{x_0}, \quad \frac{\partial\phi}{\partial y_0} = x_t x_{y_0} + y_t y_{y_0}, \tag{4}$$

$$\frac{P}{\rho} = -\frac{\partial\phi}{\partial t} - gy + \frac{1}{2}(x_t^2 + y_t^2) + \frac{\sigma}{\rho}\left[\frac{\partial^2 y}{\partial x_0^2}\frac{\partial x}{\partial x_0} - \frac{\partial^2 x}{\partial x_0^2}\frac{\partial y}{\partial x_0}\right]\cdot\left[\left(\frac{\partial x}{\partial x_0}\right)^2 + \left(\frac{\partial y}{\partial x_0}\right)^2\right]^{-3/2}, \tag{5}$$

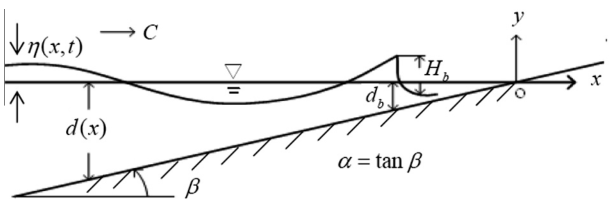

Fig. 1. Definition sketch for surface-wave propagation on a gentle sloping bottom.