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A family of hyperchaotic multi-scroll attractors in $\mathbf{R}^n$

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ABSTRACT

In this work, we present a mechanism how to yield a family of hyperchaotic multi-scroll systems in $\mathbf{R}^n$ based on *unstable dissipative systems*. This class of systems is constructed by a switching control law changing the equilibrium point of an unstable dissipative system. For each equilibrium point presented in the system a scroll emerges. The switching control law that governs the position of the equilibrium point varies according to the number of scrolls displayed in the attractor. Thus, if two systems display different numbers of scrolls, they have different switching control laws. This paper also presents a generalized theory that explains different approaches such as hysteresis and step functions from a unified viewpoint, extending the concept of chaos in $\mathbf{R}^3$ to hyperchaotic multi-scroll systems in $\mathbf{R}^n, n \geqslant 4$. An illustrative example of synchronizing a communication system is given based on the developed theory.

1. Introduction

Chaos has been an extremely well-studied area in the last decades: some recent approaches are based on suppression or induction of chaotic behavior, others focus on synchronization [1,2] and the generation and analysis of time series with the purposes of implementing them for modulation schemes or encrypting them in communication systems [3].

So far, chaotic behavior may be generated in two kinds of nonlinear systems. The first one includes continuous systems with nonlinearities given by multiplication of their states, and the second one presents a combination of piecewise-linear (PWL) systems.

The characterization [4], implementation, and design of new switched systems of chaotic behavior [5], especially possessing multiple scrolls or wings [6,7], has been of great interest for the scientific community. The methods implemented to generate multi-scroll systems in the literature may be catalogued in two: (i) systems presenting more equilibrium points than wings or scrolls, (ii) systems presenting the same number of equilibrium points and wings or scrolls.

The present study focuses on generation of a family of systems with multi-scroll attractors based only on PWL systems. This family presents three remarkable properties as follows.

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1.1. An hyperchaotic multi-scroll system.

A characterization of dynamical behavior can be achieved by means of the Lyapunov exponents. Using their values, one can determine the average exponential rates at which nearby orbits diverge or converge. Their signs define a qualitative picture of dynamics that systems may exhibit, ranging from fixed points via limit cycles and tori to more complex chaotic and hyperchaotic attractors. Whereas chaos can arise in discrete-time systems with only a single state (which must be positive), at least three state variables are required to generate chaos in continuous-time systems [8]. Such systems are characterized by one positive exponent in the Lyapunov spectrum. However, in order to obtain hyperchaos, the system must be characterized by the presence of at least two positive Lyapunov exponents. The reason is that the trajectory has to be nonperiodic and bounded within a finite region and cannot intersect itself.

Since the hyperchaotic Rössler attractor was reported in [9], many studies have been focused on generation of hyperchaotic systems [10,11], and their synchronization problem [12]. The methodology used to generate this behavior in benchmark systems, such as Lorenz and Rössler ones, is via nonlinearities of the system. The class of systems considered in this paper possesses, besides being hyperchaotic, the following property.

1.2. For each equilibrium point introduced into the system, a scroll emerges in the attractor.

It is known that the generation of multi-scroll behavior in PWL systems is based on the location of their equilibrium points. Their commutation surfaces or thresholds bound the scrolls and give a specific direction to the flow. Various papers on this topic have presented different theories developed to explain how to generate multi-scroll chaotic attractors. This paper develops an approach to generate chaotic attractors based on unstable dissipative systems.

Since the work reported by Suykens in [13] about n-double scrolls in the Chua's system, there have been different approaches to yield multi-scroll attractors. These approaches vary from modifying the nonlinear part in the Chua's system [14,15,13,16–18], to using nonsmooth nonlinear functions, such as hysteresis [19,20], saturation [21,22], threshold, and step functions [23–29]. Recently, fractional-order systems have also been used to generate multi-scroll attractors [30–32].

There have been also reports on generation of hyperchaotic multi-scroll behavior in [33–35], where the number of equilibrium points is greater than the number of scrolls. In addition to presenting hyperchaotic attractor with an equal number of scrolls and equilibrium points, the considered class of systems possesses the following third property.

1.3. The family of hyperchaotic multi-scroll systems can be extended to $\mathbf{R}^n, n \geqslant 4$.

Yalçin et al. [25] reported that 1D, 2D and 3D-grid of scrolls may be introduced locating them around the equilibrium points using a step function. Lu et al. in [19,20] presented an approach based on hysteresis that enables the creation of 1D n-scrolls, 2D $n \times m$-grid scrolls, and 3D $n \times m \times l$-grid scrolls.

In this work, continuing [29], we develop a generalized theory capable of explaining different approaches as hysteresis and step functions and extending the concept to hyperchaotic multi-scroll systems to $\mathbf{R}^n, 4 \leqslant n \in \mathbb{Z}$.

This family of systems is composed of *unstable dissipative systems* (UDS) [29,36]. Since such a system is unable to provide a stable flow by itself (due to unstable saddle points among its equilibria), a switching control law (SCL) is designed to change from one UDS to another and, by this mechanism, generate a PWL system with a multi-scroll attractor. Each scroll results from an equilibrium point and an unstable "one-spiral" trajectory it produces.

This paper is organized as follows: In Section 2, we introduce a theory explaining the generation of multi-scroll behavior via UDS. Some examples are given. Section 3 presents a family of hyperchaotic systems in $\mathbf{R}^n, n \geqslant 4$. In Section 4, we compare some of the previously known approaches to the designed UDS-based method. Section 5 presents a communication system based on the synchronization of UDS systems. Section 6 concludes this study.

2. Generation of multi-scroll attractors by UDS

We consider the class of linear system given by

$$\dot{\boldsymbol{\chi}} = \mathbf{A}\boldsymbol{\chi} + \mathbf{B} \tag{1}$$

where $\boldsymbol{\chi} = [x_1, x_2, \ldots, x_n]^T \in \mathbf{R}^n$ is the state variable, $\mathbf{B} = [B_1, B_2, \ldots, B_n]^T \in \mathbf{R}^n$ stands for a real vector, $\mathbf{A} = [\alpha_{ij}] \in \mathbf{R}^{n \times n}$ with $i, j = 1, 2, \ldots, n$ denotes a linear operator (matrix). The equilibrium point is located at $\chi^* = -\mathbf{A}^{-1}\mathbf{B}$. The system dynamic is given by the matrix $\mathbf{A}$, which has a stable manifold E^s and an unstable one E^u. On the basis of the previous discussion, it is possible to define two types of UDS as follows:

Definition 2.1. We say that system (1) is an UDS of Type I if $\sum_{i=1}^{n} \lambda_i < 0$, where $\lambda_i, i = 1, \ldots, n$, are eigenvalues of $\mathbf{A}$, and at least one λ_i is a negative real eigenvalue, and at least two λ_i are complex conjugate eigenvalues with positive real part $Re\{\lambda_i\} > 0$.

Definition 2.2. We say that system (1) is an UDS of Type II if $\sum_{i=1}^{n} \lambda_i < 0$, where $\lambda_i, i = 1, \ldots, n$, are eigenvalues of $\mathbf{A}$, at least one λ_i is a positive real eigenvalue, and at least two λ_i are complex conjugate eigenvalues with negative real part $Re\{\lambda_i\} < 0$.