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Interaction between Hermitian and normal imbeddings

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ABSTRACT

For several applications, it is highly desirable to understand how the eigenvalues of an imbeddable matrix V^*AV, where $V \in \mathbb{C}^{n\times k}$ is an isometry, are distributed throughout the numerical range of $A \in \mathbb{C}^{n\times n}$. There has been extensive study for A Hermitian, while a geometric description for the eigenvalues of imbeddings in non-Hermitian matrices remains a challenging problem. Toward this direction, a subspace is introduced, wherein all $n \times n$ complex diagonal matrices for which a given isometry $V \in \mathbb{C}^{n\times k}$ generates diagonal imbeddings are defined. In particular, conditions upon which a real diagonal matrix may be imbeddable in some normal are obtained, including an application for higher rank numerical ranges. Finally, a procedure determining whether two given sets of complex numbers may be realized as spectra of a pair of imbeddable normal matrices is established.

1. Introduction

For a pair of matrices $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{k\times k}$ $(1 \leqslant k < n)$, we shall call B *imbeddable* in A if there exists an isometry $V \in \mathbb{C}^{n\times k}$ (i.e. $V^*V = I_k$) such that $V^*AV = B$. In the following, V will be referred to as *generating isometry* for B in A, while the imbedding relation above will be denoted by $B \overset{V}{\sim} A$. The special cases where A and B are both Hermitian or normal have been the subject of extensive study and particular emphasis has been placed on relating the spectra of the matrices involved. For instance, a necessary imbedding condition regarding the arguments of the eigenvalues of A and B appeared in [6], while in [12] it was shown that these are interlacing with respect to the lexicographic orders in $\mathbb{C}$. Links on several imbedding conditions which have appeared in the literature can be found in [9]. Of particular interest is the case $k = n - 1$, in which a simplification occurs, due to the fact that every $(n-1)$-dimensional subspace $\mathcal{S}$ of $\mathbb{C}^n$ is uniquely determined by a nonzero vector v satisfying $V^*v = 0$, where V is an $n \times (n-1)$ isometry with range $\mathrm{R}(V) = \mathcal{S}$ (see [5,7] for A, B both Hermitian or normal and [10,11,2] for A normal and B arbitrary).

Recall that the *numerical range* of a matrix $A \in \mathbb{C}^{n\times n}$ is a closed, convex subset of the complex plane defined by

$$w(A) = \{x^*Ax : x \in \mathbb{C}^n, \|x\| = 1\}$$

and when $A \in \mathbb{C}^{n\times n}$ is normal, $w(A)$ coincides with the convex hull of its eigenvalues, i.e. $w(A) = \mathrm{co}\{\sigma(A)\}$. An inverse problem for $w(A)$ was recently introduced in [7], where, given a Hermitian or normal matrix $A \in \mathbb{C}^{n\times n}$ and a set of points $\mu_1, \ldots, \mu_k \in w(A)$, it is undertaken to determine an isometry $V \in \mathbb{C}^{n\times k}$, such that $\mathrm{diag}\{\mu_1, \ldots \mu_k\} \overset{V}{\sim} A$. In the context of iterative methods for solving eigenvalue problems, the diagonal entries of V^*AV are referred to as *Ritz values* of A with respect to the isometry V and, under this scope, a problem closely related to the one above has been stated and studied in [1,2] respectively. In the restarted Arnoldi method for eigenvalue computation [13], Ritz values play an important role both as

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eigenvalue estimates and as the roots of restart polynomials. Hence, the study of imbeddings could provide a better understanding of Ritz value properties and distribution throughout $w(A)$, which is necessary for the analysis of the convergence of such algorithms.

In the special case that the imbeddable matrix is *scalar*, the concept of imbedding is related to *higher rank numerical ranges*,

$$\begin{aligned}\Lambda_k(A) &= \{\lambda \in \mathbb{C} : PAP = \lambda P,\ \text{for some rank} - k \text{ projection } P\}\\ &= \{\lambda \in \mathbb{C} : \lambda I_k \text{ is imbeddable in } A\}.\end{aligned}$$

The higher rank numerical range has found numerous applications in *quantum error correction*. Especially, for the construction of error correcting codes in quantum computing, it is crucial to obtain generating isometries corresponding to points in higher rank numerical ranges [4].

We remark that for $A \in \mathbb{C}^{n\times n}$ normal and $1 \leqslant k < n$ some given dimension, in general there does not exist a $k \times k$ imbeddable diagonal matrix. A necessary and sufficient condition for the existence of isometries $V \in \mathbb{C}^{n\times k}$ that lead to diagonal imbeddings in an $n \times n$ normal matrix is established in [8], whereby the columns $\{v_\ell\}_{\ell=1}^k$ of V constitute a set of k mutually orthogonal and A-orthogonal vectors (i.e. $v_i^* v_j = v_i^* A v_j = v_i^* A^* v_j = 0$, for $i \neq j \in \{1,\ldots,k\}$) and are such that

$$\dim\mathcal{T}_\ell \equiv \dim\{\operatorname{Ker}(V_\ell^*) \cap \operatorname{Ker}(V_\ell^* A) \cap \operatorname{Ker}(V_\ell^* A^*)\} \geqslant k - \ell \tag{1}$$

for $V_\ell = [\, v_1 \quad \cdots \quad v_\ell \,]$ $(\ell = 1,\ldots,k-1)$. In this case, a unit vector $v_{\ell+1} \in \mathcal{T}_\ell$ exists and we may let the $n \times (\ell+1)$ isometry $V_{\ell+1} = [\, V_\ell \quad v_{\ell+1} \,]$ that generates a diagonal matrix of order $\ell+1$. Hence, for $\ell = k-1$, the isometry $V \equiv V_k$ is generating for a diagonal matrix of $\mathbb{C}^{k\times k}$ in A. As presented in [8, Theorem 1], condition (1) is always true for integers $\ell \in [1, \frac{n-1}{3}]$ and for $\ell = k-1$ we should have $k \leqslant \frac{n+2}{3}$. Additionally, for the pairs of isometries V_ℓ and U_ℓ $(\ell = 1,\ldots,k-1)$ with $\mathrm{R}(U_\ell) = \operatorname{Ker}(V_\ell^*)$, the vector $v_{\ell+1} \in \mathcal{T}_\ell$ may be expressed as $v_{\ell+1} = U_\ell y_\ell$, where $y_\ell \in \operatorname{Ker}\begin{bmatrix} V_\ell^* A U_\ell \\ V_\ell^* A^* U_\ell \end{bmatrix}$.

Despite the fact that the notion of imbedding has been the subject of study for several authors, the connection between pairs of diagonal imbeddings in Hermitian and normal matrices has not been investigated elsewhere. Motivated by the procedure in [8] outlined above, in this paper we examine further properties of isometries with mutually A-orthogonal columns. These allow us to study the interplay between Hermitian and normal imbeddings, which turn out to be intimately related. Since Hermitian and normal matrices are unitarily diagonalizable, throughout this paper we may assume without loss of generality that the matrices under consideration are either real or complex diagonal. Indeed, considering the diagonalized form of $A = UDU^*$ with $U \in \mathbb{C}^{n\times n}$ unitary, then the isometry V generates a diagonal imbedding in A if and only if the same holds for the isometry U^*V in D.

In the next section, we present conditions upon which an isometry V is simultaneously generating for diagonal imbeddings in a pair of normal and Hermitian matrices A and B respectively. Toward this direction, given an isometry V, we introduce a subspace $\mathcal{S}$ of $\mathbb{C}^n$ defining all possible diagonal matrices in $\mathbb{C}^{n\times n}$, in which V generates diagonal imbeddings (Theorem 1). Our approach relies intimately on the properties of $\mathcal{S}$. The following section examines the case where a normal matrix may have a real diagonal imbedding. In particular, given a real diagonal matrix $C \in \mathbb{R}^{k\times k}$, a corresponding normal matrix A is constructed (Theorem 2), so that C is imbeddable in A. Moreover, an application for higher rank numerical ranges is considered as well, while conditions on the columns of V are stated, upon which, given a pair of complex and real diagonal matrices A, B respectively, the diagonal imbeddings V^*AV and V^*BV coincide (Proposition 3). In the final section, our analysis leads to the determination of sufficient imbedding conditions, concluding whether two given sets of complex numbers can be realized as spectra of a pair of imbeddable normal matrices.

2. Space for diagonal imbeddings

In this section, elaborating further on ideas from [8], we discuss conditions upon which an isometry V is simultaneously generating for real or complex diagonal imbeddings in a pair of Hermitian and normal matrices respectively. Hence, we demonstrate when for a normal matrix $A \in \mathbb{C}^{n\times n}$ and an isometry $V \in \mathbb{C}^{n\times k}$, such that V^*AV is diagonal in $\mathbb{C}^{k\times k}$, there exists a corresponding Hermitian $B \in \mathbb{C}^{n\times n}$, for which $V^*BV \in \mathbb{R}^{k\times k}$ is also diagonal and vice versa. Clearly, it is immediate that for B we may consider the Hermitian parts, $\mathrm{H}(A) = \frac{A+A^*}{2}$ or $\mathrm{S}(A) = \frac{A-A^*}{2\mathbf{i}}$, of A and even all Hermitian matrices arising as real shifts of any $B \in \operatorname{span}\{\mathrm{H}(A), \mathrm{S}(A)\} \cap \mathbb{R}^{n\times n}$. Therefore, we are interested in determining criteria for the existence of other Hermitian matrices B, for which V generates a diagonal imbedding as well.

Furthermore, the converse problem is also considered, i.e. when for Hermitian B and an isometry $V \in \mathbb{C}^{n\times k}$ such that V^*BV is real diagonal, we should define a normal matrix A, for which V^*AV is also diagonal. Since any $A \in \operatorname{span}\{B, I_n\}$ has this property as well and noticing that its numerical range $w(A)$ is related to the line segment $w(B)$ via an affine transformation, we focus hereafter on the construction of normal matrices A whose numerical range has nonempty interior.

As noted in the introduction, since normal matrices are unitarily diagonalizable, the matrix A may be taken to be diagonal and our main tool states a necessary and sufficient condition, so that V^*AV is also diagonal. In the following, according to the familiar Matlab convention, from a vector $x = (x_j)_{j=1}^n \in \mathbb{C}^n$ we define a corresponding diagonal $n \times n$ matrix $\mathtt{diag}(x) = \operatorname{diag}\{x_j\}_{j=1}^n$ and vice versa, i.e. for $X = \operatorname{diag}\{x_j\}_{j=1}^n$ we denote $\mathtt{diag}(X) = (x_j)_{j=1}^n$ its associated vector.