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Harmonic shears of slit and polygonal mappings ☆

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ABSTRACT

In this paper, we study harmonic mappings by using the *shear construction*, introduced by Clunie and Sheil-Small in 1984. We consider two classes of conformal mappings, each of which maps the unit disk $\mathbb{D}$ univalently onto a domain which is convex in the horizontal direction, and shear these mappings with suitable dilatations ω. Mappings of the first class map the unit disk $\mathbb{D}$ onto four-slit domains and mappings of the second class take $\mathbb{D}$ onto regular n-gons. In addition, we discuss the minimal surfaces associated with such harmonic mappings. Furthermore, illustrations of mappings and associated minimal surfaces are given by using MATHEMATICA.

1. Introduction

A complex-valued harmonic function f defined on the unit disk $\mathbb{D}$ is called a *harmonic univalent mapping* if it maps $\mathbb{D}$ univalently onto a domain $\Omega \subset \mathbb{C}$. Note that it is not required that the real and the imaginary part of f satisfy the Cauchy–Riemann equations. In 1984, Clunie and Sheil-Small [3] showed that many classical results for conformal mappings have natural analogues for harmonic mappings, and hence they can be regarded as a generalization of conformal mappings. Each harmonic mapping in $\mathbb{D}$ has a canonical presentation $f = h + \overline{g}$, where h and g are analytic in $\mathbb{D}$ and $g(0) = 0$. A harmonic mapping $f = h + \overline{g}$ is called *sense-preserving* if the Jacobian $J_f = |h'|^2 - |g'|^2$ is positive in $\mathbb{D}$. Then f has an *analytic dilatation* $\omega = g'/h'$ such that $|\omega(z)| < 1$ for $z \in \mathbb{D}$. For basic properties of harmonic mappings we refer to [4,9,14].

A domain $\Omega \subset \mathbb{C}$ is said to be *convex in the horizontal direction* (CHD) if its intersection with each horizontal line is connected (or empty). A univalent harmonic mapping is called a CHD mapping if its range is a CHD domain. Construction of a harmonic mapping f with prescribed dilatation ω can be done effectively by the *shear construction* the devised by Clunie and Sheil-Small [3].

Theorem 1.1. *Let $f = h + \overline{g}$ be a harmonic and locally univalent in the unit disk $\mathbb{D}$. Then f is univalent in $\mathbb{D}$ and its range is a CHD domain if and only if $h - g$ is a conformal mapping of $\mathbb{D}$ onto a CHD domain.*

Suppose that φ is a CHD conformal mapping. For a given dilatation ω, the harmonic shear $f = h + \overline{g}$ of φ is obtained by solving the differential equations

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$$\begin{cases} h' - g' &= \varphi', \\ \omega h' - g' &= 0. \end{cases}$$

From the above equations, we obtain

$$h(z) = \int_0^z \frac{\varphi'(\zeta)}{1 - \omega(\zeta)}\, d\zeta. \tag{1}$$

For the anti-analytic part g, we have

$$g(z) = \int_0^z \omega(\zeta) \frac{\varphi'(\zeta)}{1 - \omega(\zeta)}\, d\zeta. \tag{2}$$

Observe that

$$f(z) = h(z) + \overline{g(z)} = 2\,\mathrm{Re}\left[\int_0^z \frac{\varphi'(\zeta)}{1 - \omega(\zeta)}\, d\zeta\right] - \overline{\varphi(z)}. \tag{3}$$

We shall use (1) to find the analytic part h of the harmonic mapping $f = h + \overline{g}$. Then the anti-analytic part g of the harmonic mapping f can be obtained from the identity $g = h - \varphi$, or computed via (3). A step-by-step algorithm can be given as follows:

Algorithm 1.2. (Harmonic Shearing)

1. Choose a CHD conformal mapping φ.
2. Choose a dilatation ω.
3. Compute h and g via (1) and (2), respectively.
4. Construct the harmonic mapping $f = h + \overline{g}$.

It is known that the class of harmonic mappings has a close connection with the theory of minimal surfaces. In the space $\mathbb{R}^3$, a *minimal surface* is a surface which minimizes the area with a fixed curve as its boundary. This minimization problem is called *Plateau's Problem*. Discussion concerning the differential geometric approach to the subject can be found from the book by Pressley [15].

Our results concerning minimal surfaces are based on the Weierstrass-Enneper representation. Let S be a non-parametric minimal surface over a simply connected domain Ω in $\mathbb{C}$ given by

$$S = \{(u, v, F(u, v)) : u + iv \in \Omega\},$$

where (u, v) identifies the complex plane $\mathbb{R}^2$, which underlies the domain of F. The following result is known as the Weierstrass-Enneper representation. This representation provides a link between harmonic univalent mappings and minimal surfaces. The surface S is a minimal surface if and only if S has the following representation

$$S = \left\{ \left(\mathrm{Re} \int_0^z \varphi_1(\zeta)\, d\zeta + c_1,\ \mathrm{Re} \int_0^z \varphi_2(\zeta)\, d\zeta + c_2,\ \mathrm{Re} \int_0^z \varphi_3(\zeta)\, d\zeta + c_3 \right) : z \in \mathbb{D} \right\},$$

where $\varphi_1, \varphi_2, \varphi_3$ are analytic such that $\varphi_1^2 + \varphi_2^2 + \varphi_3^2 = 0$, and

$$f(z) = u(z) + iv(z) = \mathrm{Re} \int_0^z \varphi_1(z)\, dz + i\mathrm{Re} \int_0^z \varphi_2(z)\, dz + c$$

is a sense-preserving univalent harmonic mapping from $\mathbb{D}$ onto Ω. In this case, the surface S is called a minimal graph over Ω with the projection $f = u + iv$. Further information about the relation between harmonic mappings and minimal surfaces can be found from [5,9].

Systematical construction of harmonic shears of mappings of the unit disk and unbounded strip domains, and their boundary behaviour are presented in the article by Greiner [11]. In most cases the dilatation is chosen to be $\omega(z) = z^n$. In this paper, we study two classes of conformal mappings, each of which map $\mathbb{D}$ univalently onto a domain which is convex in the horizontal direction. The first one involves the mapping

$$\varphi(z) = A \log\left(\frac{1+z}{1-z}\right) + B \frac{z}{1 + cz + z^2},$$

which map $\mathbb{D}$ onto $\mathbb{C}$ minus four symmetric half-lines. In [10], Ganczar and Widomski have studied some special cases of this mapping and its harmonic shears. Analytic examples of harmonic shears of φ with dilatations $\omega(z) = \pm z^2$ and $\omega(z) = -z^4$, along with illustrations, are given in [6].