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Linearly localized difference schemes for the nonlinear Maxwell model of a magnetic field into a substance ☆

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ABSTRACT

A linearly localized difference scheme with the first-order time approximation, is proposed for solving a nonlinear Maxwell model associated with the penetration of a magnetic field into a substance. The new scheme is computationally efficient since the resulting algebra equations are linear and can be computed by the fast Thomas algorithm without any Newton-type inner iterations. It is also local in time, that is, only numerical solutions in one previous time-level are necessary to update the current solutions, such that it requires much less storage compared with the fully implicit method. Furthermore, the exponential decaying behavior of difference solution, which is analogous to that of the continuous solution, is obtained. To improve the time accuracy, we apply the Crank–Nicolson-type time discretization to construct a second-order linearly localized method. Numerical examples are presented to support our theoretical results.

1. Introduction

We propose two linearly localized difference methods for solving the nonlinear Maxwell equations arising in mathematical modeling of the process of a magnetic field penetrating into a substance. A variable magnetic field induces in the material a variable electronic field which causes the appearance of currents. The currents lead to the heating of the material and elevating its temperature. Consider the quasi-stationary approximation, the corresponding system of Maxwell's equations has the form

$$\frac{\partial \mathbf{H}}{\partial t} = -\nabla \times (v_s \nabla \times \mathbf{H}), \quad c_s \frac{\partial T_s}{\partial t} = v_s |\nabla \times \mathbf{H}|^2, \tag{1.1}$$

where $\mathbf{H} = (H_x, H_y, H_z)^T$ is a vector of the magnetic field, T_s is temperature, and c_s and v_s are the thermal heat capacity and electro-conductivity of the substance. If c_s and v_s of the substance highly dependent on temperature, then the Maxwell's system can be rewritten in the following form [2]

$$\frac{\partial \mathbf{H}}{\partial t} = \nabla \times \left[a\left(\int_0^t |\nabla \times \mathbf{H}|^2 \mathrm{d}\tau \right) \nabla \times \mathbf{H} \right], \tag{1.2}$$

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where the function $a = a(\sigma)$ is defined for $\sigma \geqslant 0$. Let $\mathbf{H} = (0, 0, U)^T$ with a scalar function $U = U(x, t)$, the above equation becomes a one-dimensional nonlinear diffusion equation

$$U_t = \frac{\partial}{\partial x}\left[a(S)\frac{\partial U}{\partial x}\right], \tag{1.3}$$

where the nonlinear coefficient

$$S(x, t) = \int_0^t |U_x(x, \tau)|^2 \mathrm{d}\tau. \tag{1.4}$$

Take the magnetic field $\mathbf{H} = (0, U, V)^T$ in (1.2) with the scalar functions $U = U(x, t)$ and $V = V(x, t)$, Eq. (1.2) becomes the nonlinear system of two integro-differential equations

$$U_t = \frac{\partial}{\partial x}\left[a(Q)\frac{\partial U}{\partial x}\right], \quad V_t = \frac{\partial}{\partial x}\left[a(Q)\frac{\partial V}{\partial x}\right], \tag{1.5}$$

where the nonlinear coefficient

$$Q(x, t) = \int_0^t \left(|U_x(x, \tau)|^2 + |V_x(x, \tau)|^2\right)\mathrm{d}\tau. \tag{1.6}$$

Note that the nonlocal Eqs. (1.2), (1.3) and (1.5) are complex and some special cases of such type models were only investigated, for example, in [2–5]. The existence and uniqueness of global solution for the initial-boundary value problem have been proven in [3]. The asymptotic behavior of the solutions to the initial-boundary value problem for (1.3) and (1.5)-type models have also been investigated recently, see [7–10].

Numerical methods for the nonlinear integro-differential equations were investigated in [6,11–13]. Finite element approximation and finite difference scheme for one nonlinear model arising from one-dimensional heat flow in materials with memory have been proposed in [12,13]. Neta developed Galerkin finite element method in [12], and Neta and Igwe [13] compared the finite element solution in [12] with an explicit difference approximation from the point of view of accuracy and computer storage. It has been shown that the finite difference method yields comparable results for the same mesh spacing using less computer storage. Taking the simple coefficient $a(s) = 1 + s$, Jangveladze et al. [6] proposed the following implicit difference scheme (see the discrete notations in Section 3) to approximate the Eq. (1.3),

$$\frac{u_i^{n+1} - u_i^n}{\tau} = \delta_x\left[\left(1 + \tau\sum_{k=0}^{n+1}\left|\delta_x u_{i-\frac{1}{2}}^k\right|^2\right)\delta_x u_{i-\frac{1}{2}}^{n+1}\right], \quad n \geqslant 0. \tag{1.7}$$

It has been shown that the fully implicit difference scheme (1.7) is convergent with an order of $O(\tau + h)$. Recently, this scheme is generalized in [11] to solve one-dimensional nonlinear systems (1.5).

Note that, the fully implicit difference scheme (1.7) and its generalization in [11] are nonlinear and computationally expensive because a Newton-type iteration [1] is always necessary to solve a large system of nonlinear equations at each time level, see e.g. [11]. Although the nonlinear implicit schemes are stable and admit large time-steps, the convergence of the Newton's iteration may be slow especially when large time-steps are used. However, linearly implicit schemes are attractive because the computational cost is roughly the same at each time level. In this report, we suggest two linearized difference algorithms, one is first-order and another is second-order in time, which are shown to be computationally efficient. Furthermore, the present methods are local in time, that is, only numerical solutions in one previous time-level are necessary to update the current solutions. For the first-order linearized scheme, we obtain the long-time exponential decaying behavior of the discrete solution, and apply the discrete energy method to prove the solvability and stability. Under a reasonable assumption on the discrete solution, it is shown that the difference solution is convergent with an order of $O(\tau + h^2)$. To improve the time accuracy, we apply the Crank–Nicolson scheme to construct another linearized algorithm which is experimentally convergent with an order of $O(\tau^2 + h^2)$.

The content of this report will be organized as follows. In the next section, long-time decaying behaviors of continuous solution for the initial-boundary value problem are recalled. Section 3 presents the first-order linearized scheme and shows that the discrete solution has the asymptotic property analogous to that of the continuous solution. The solvability, stability and convergence of the first-order difference method are addressed in Section 4. To improve the time accuracy, a Crank–Nicolson-type approximation is employed in Section 5 to construct a second-order linearized difference scheme. Numerical experiments are presented in Section 6 to support our analysis. Short comments including some open problems conclude the article.

2. Long-time behavior of continuous solution

Consider the following initial-boundary value problem

$$U_t = \frac{\partial}{\partial x}\left[(1 + S)\frac{\partial U}{\partial x}\right] + f(x, t), \quad x \in \Omega,\ t \in (0, \infty), \tag{2.1}$$