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Julia sets and Mandelbrot sets in Noor orbit

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ABSTRACT

In recent literature, researchers have generated Julia sets and Mandelbrot sets in Mann and Ishikawa orbits that are examples of two-step and three-step feedback processes respectively. This paper presents further generalization of Julia and Mandelbrot sets for complex-valued polynomials such as quadratic, cubic and higher degree polynomials using a Noor orbit, which is a four-step iterative procedure. The graphical images of Julia and Mandelbrot sets have been visualized and certain patterns in Mandelbrot sets have been recognized. It is fascinating to see that a few Mandelbrot sets are akin to a butterfly or a coupled urn or a coupled trident.

1. Introduction

With the introduction of fractal geometry, mathematics has presented some interesting complex objects to computer graphics. Interest in Julia sets and related mathematics began in 1920's with Gaston Julia [28, p. 122]. What makes Julia sets interesting to study is that despite being born out of apparently simple iterative processes they can be very intricate and often fractal in nature [28, p. 122]. In 1975, Benoit Mandelbrot extended the work of Gaston Julia and introduced the Mandelbrot set; a set of all connected Julia sets. Mandelbrot and Julia sets have been studied for quadratic [12,13,22,28], cubic [8,9,13–16,23] and higher degree polynomials [17], under Picard orbit, which is an example of one-step feedback process.

Julia and Mandelbrot sets have been studied under the effect of noises [2–7,42,43,45,47,48,51] arising in the objects. Rochon [39] studied a more generalized form of a Mandelbrot set, which lives in a bi-complex plane (see also [18,41,50]). Later on, in a series of papers, Wang and Chang [44], Wang et al. [46], Wang and Luo [49], Wang et al. [52] jointly with others carried further analysis of generalized Julia and Mandelbrot sets. In 2006, they generalized the Lyapunov exponents and periodic scanning techniques given by Wang and Chang [44], Wang et al. [52] and put forward periodicity orbit search and comparison technique, which can be used to discuss the relationship of the generalized Mandelbrot and Julia sets. Also, they studied the fractal structure and discontinuity evolution law of the generalized Julia sets generated from the extended complex mapping $z^n - c\,(n \in R)$ [49]. In 2008, they studied on the real part of the map $z^n + c\,(n > 1)$ and also analyzed the dynamics of a family of one-dimensional maps [53]. Further, they presented the periodic regions centers in the general M-sets with integer index number [46].

In 2004, Rani and Kumar [36,37] introduced superior iterates (a two-step feedback process) in the study of fractal theory, and created superior Julia and Mandelbrot sets. Later on, in a series of papers Rani, jointly with other researchers, generated and analyzed superior Julia and superior Mandelbrot sets for quadratic [21,34,38], cubic [29], and nth degree [30,31,35] complex polynomials. After creation of superior Mandelbrot sets, Negi and Rani [25] collected the properties of midgets

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of quadratic superior Mandelbrot sets. Negi et al. [26] simulated the behavior of Julia sets using switching processes. Superior Julia and superior Mandelbrot sets have also been studied under the effect of noises [1,24,32,33]. For a complete literature review of superior fractals, one may refer to Singh et al. [40]. Recently, Chauhan et al. [10,11] have obtained new Julia and Mandelbrot sets via Ishikawa iterates (an example of three-step feedback process).

In this paper, we obtain new Julia and Mandelbrot sets via the Noor iterative procedure, which is an example of a four-step feedback process. In Section 2, we define the Noor iterative procedure, which is the basis of our work. In Section 3, we discuss escape criterions for quadratic, cubic and nth degree polynomials under the Noor orbit. Several Julia and Mandelbrot sets have been presented in Sections 4 and 5 respectively. Finally, the paper has been concluded in Section 6.

2. Preliminaries

Definition 1 (*Picard orbit*). Let X be a non-empty set and $f: X \to X$. For a point x_0 in X, the Picard orbit (generally called the orbit of f) is the set of all iterates of a point x_0, i.e.,

$$O(f, x_0) = \{x_n : x_n = fx_{n-1},\ n = 1, 2, \ldots\},$$

where the orbit $O(f,x_0)$ of f at the initial point x_0 is the sequence $\{f^n x_0\}$.

Definition 2 (*Julia set*). The filled in Julia set of the function Q is defined as

$$K(Q) = \{z \in C : Q^k(z) \text{ does not tend to } \infty\},$$

where C is the complex space, $Q^k(z)$ is kth iterate of function Q and $K(Q)$ denotes the filled Julia set. The Julia set of the function Q is defined to be the boundary of $K(Q)$, i.e.,

$$J(Q) = \partial K(Q),$$

where $J(Q)$ denotes the Julia set. The set of points whose orbits are bounded under the Picard orbit $Q_c(z) = z^2 + c$ is called the Julia set. We choose the initial point 0, as 0 is the only critical point of Q_c [13, p. 225].

Definition 3 (*Mandelbrot set*). The Mandelbrot set M consists of all parameters c for which the filled Julia set of Q_c is connected, that is

$$M = \{c \in C : K(Q_c) \text{ is connected}\}.$$

In fact, M contains an enormous amount of information about the structure of Julia sets. The Mandelbrot set M for the Quadratic $Q_c(z) = z^2 + c$ is defined as the collection of all $c \in C$ for which the orbit of the point 0 is bounded, that is

$$M = \{c \in C : \{Q_c^n(0)\};\ n = 0, 1, 2, \ldots \text{ is bounded}\}.$$

We choose the initial point 0 as 0 is the only critical point of Q_c [13, p. 249].

Definition 4 (*Noor orbit*). Let us consider a sequence $\{x_n\}$ of iterates for initial point $x_0 \in X$ such that

$$\{x_{n+1} : x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n;\ y_n = (1-\beta_n)x_n + \beta_n T z_n;\ z_n = (1-\gamma_n)x_n + \gamma_n T x_n;\ n = 0, 1, 2, \ldots\},$$

where $\alpha_n, \beta_n, \gamma_n \in [0,1]$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences of positive numbers. The above sequence of iterates is called as Noor orbit, denoted by *NO,* which is a function of five tuple (T, x_0, α_n, β_n, γ_n) [27].

Notice that the *NO* reduces to:

1. The Ishikawa orbit when $\gamma_n = 0$;
2. The Mann orbit when $\beta_n, = \gamma_n = 0$; and
3. The Picard orbit *when* $\beta_n, = \gamma_n = 0$ and $\alpha_n = 1$.

All that follows, for the sake of simplicity, we take $\alpha_n = \alpha$, $\beta_n = \beta$ and $\gamma_n = \gamma$.

3. Escape criterions for complex polynomials in the Noor orbit

The escape criterion plays a vital role in the generation and analysis of Julia sets, Mandelbrot sets and their variants. We need the following escape criterions for the quadratics, cubic and higher degree polynomials.

3.1. Escape criterions for quadratic polynomials

Throughout this section, we assume that $T(x_n) = Q_c''(z)$, $T(z_n) = Q_c'(z)$ and $T(y_n) = Q_c(z)$.