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Classification of neutral difference equations of any order with respect to the asymptotic behavior of their solutions

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ABSTRACT

The asymptotic behavior of the solutions to a neutral difference equation of the form

$$\Delta^m[x(n) + q(n)x(\tau(n))] + p(n)x(\sigma(n)) = 0, \quad m \in \mathbb{N}, \quad n \geqslant 0,$$

where $\tau(n)$ is a general retarded argument, $\sigma(n)$ is a general deviated argument, $(p(n))_{n\geqslant 0}$ and $(q(n))_{n\geqslant 0}$ are sequences of real numbers, Δ denotes the forward difference operator $\Delta x(n) = x(n+1) - x(n)$, and Δ^j denotes the jth forward difference operator $\Delta^j(x(n)) = \Delta\left(\Delta^{j-1}(x(n))\right)$ for $j = 2, 3, \ldots, m$, is studied. Examples illustrating the results are also given.

1. Introduction

Neutral difference equations are difference equations in which the higher order difference of the unknown sequence appears in the equations both with and without delays or advances.

Besides the theoretical interest, the study of the asymptotic and oscillatory behavior of the solutions of neutral type equations is motivated by their application in several areas of applied mathematics. Neutral type (difference or differential) equations arise in circuit theory, bifurcation analysis, population dynamics, signal processing and several other fields. For the general theory of difference equations the reader is referred to the monographs [1,13,18].

In the present paper, we study the asymptotic behavior of the solutions of a first-order neutral difference equation (1st-order NDE) of the form

$$\Delta[x(n) + q(n)x(\tau(n))] + p(n)x(\sigma(n)) = 0, \quad n \geqslant 0, \tag{E$_1$}$$

where $(p(n))_{n\geqslant 0}$ and $(q(n))_{n\geqslant 0}$ are sequences of real numbers, $(\tau(n))_{n\geqslant 0}$ is an increasing sequence of integers which satisfies

$$\tau(n) \leqslant n - 1, \quad \forall n \geqslant 0 \quad \text{and} \quad \lim_{n\to\infty} \tau(n) = +\infty \tag{1.1}$$

and $(\sigma(n))_{n\geqslant 0}$ is an increasing sequence of integers such that either $\sigma(n) \leqslant n - 1$, $\forall n \geqslant 0$ and $\lim_{n\to\infty} \sigma(n) = +\infty$ or $\sigma(n) \geqslant n + 1$, $\forall n \geqslant 0$.

Next, we study the asymptotic behavior of the solutions of a higher-order neutral difference equation (mth-order NDE) of the form

$$\Delta^m[x(n) + q(n)x(\tau(n))] + p(n)x(\sigma(n)) = 0, \quad \mathbb{N} \ni m \geqslant 2, \ n \geqslant 0. \tag{E$_m$}$$

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Define

$$k = -\min\{\tau(0), \sigma(0)\} \quad \text{if } \sigma(n) \text{ is a retarded argument.}$$

(Clearly, k is a positive integer.)

By a *solution* of (E$_1$) or (E$_m$), we mean a sequence of real numbers $(x(n))_{n\geqslant -k}$ which satisfies (E$_1$) or (E$_m$) for all $n \geqslant 0$. It is clear that, for each choice of real numbers $c_{-k}, c_{-k+1}, \ldots, c_{-1}, c_0$, there exists a unique solution $(x(n))_{n\geqslant -k}$ of (E$_1$) or (E$_m$) which satisfies the initial conditions $x(-k) = c_{-k},\ x(-k+1) = c_{-k+1}, \ldots,\ x(-1) = c_{-1},\ x(0) = c_0$.

If $\sigma(n)$ is an advanced argument, then $k = -\tau(0)$ and by a *solution* of (E$_1$) or (E$_m$), we mean a sequence of real numbers $(x(n))_{n\geqslant -k}$, which satisfies (E$_1$) or (E$_m$) for all $n \geqslant 0$.

A solution $(x(n))_{n\geqslant -k}$ of (E$_1$) or (E$_m$) is called *oscillatory*, if the terms $x(n)$ of the sequence are neither eventually positive nor eventually negative. Otherwise, the solution is said to be *nonoscillatory*.

In the last few decades, the asymptotic and oscillatory behavior of neutral difference equations has been extensively studied. See, for example [2–12,14–17,19–27] and the references cited therein. Most of these papers concern the special case where the delay $(n - \tau(n))_{n\geqslant 0}$ is constant, while a small number of these papers are dealing with the general case of Eq. (E$_1$) or (E$_m$), in which the delay $(n - \tau(n))_{n\geqslant 0}$ is variable. The aim in this paper is to study the asymptotic behavior of the solutions of neutral difference equations with variable delays of the general form of Eq. (E$_1$) or (E$_m$). Examples illustrating the results are also presented.

2. Some preliminaries

Define the notation

$$\tau \circ \tau = \tau^2, \quad \tau \circ \tau \circ \tau = \tau^3, \quad \text{and so on.} \tag{2.1}$$

Let the domain of τ be the set $D(\tau) = \mathbb{N}_{n_*} = \{n_*, n_* + 1, n_* + 2, \ldots\}$, where $n_* \geqslant 0$ is the smallest natural number that τ is defined. Then for every $n > n_*$ there exists a natural number $m(n)$ such that

$$\tau^{m(n)}(n) = \tau(n_\lambda) \quad \text{and} \quad \lim_{n\to\infty} m(n) = +\infty, \tag{2.2}$$

since $(m(n))$ is an increasing and unbounded function of n where $n_\lambda \geqslant n_*,\ \tau(n_\lambda) \leqslant n_*$ and $n_\lambda = \tau^{m(n)-1}(n)$.

Assume that $\limsup_{n\to\infty} q(n) = c$ and $\liminf_{n\to\infty} q(n) = d$. We are going to use the following conditions:

$$c < -1, \tag{C$_1$}$$

$$c - d < 1, \tag{C$_2$}$$

$$\lim_{n\to\infty} q(n) = -1, \tag{C$_3$}$$

$$\lim_{n\to\infty} \prod_{i=0}^{m(n)} \left(-q(\tau^i(n))\right) = B < +\infty, \tag{C$_4$}$$

$$-1 < q(n) < 0 \quad \text{and} \quad d > -1. \tag{C$_5$}$$

We define the sequence $(z(n))$ as follows

$$z(n) := x(n) + q(n)x(\tau(n)), \tag{2.3}$$

where $(x(n))$ is a solution of (E$_1$) or (E$_m$) or a given well-defined sequence of real numbers. Below we will often assume that $(z(n))$ tends to $\pm\infty$ or that $\lim_{n\to\infty} z(n)$ exists as a real number, i.e., $\lim_{n\to\infty} z(n) = A \in \mathbb{R}$.

The following lemmas provide useful tools for establishing the main results:

Lemma 2.1. *Assume that $(x(n))$ is a nonoscillatory solution of (E$_1$) or (E$_m$). Then the following statements hold:*

(i) *If $p(n) \geqslant 0,\ \forall n \geqslant 0$ then $(z(n))$ is eventually nonincreasing when $(x(n))$ is a solution of (E$_1$) while $\Delta^m z(n) \leqslant 0$ when $(x(n))$ is a solution of (E$_m$).*
(ii) *If $p(n) \leqslant 0,\ \forall n \geqslant 0$ then $(z(n))$ is eventually nondecreasing when $(x(n))$ is a solution of (E$_1$) while $\Delta^m z(n) \geqslant 0$ when $(x(n))$ is a solution of (E$_m$).*

Proof. Part (i): Assume that the solution $(x(n))_{n\geqslant -k}$ of (E$_1$) or (E$_m$) is nonoscillatory. Then it is either eventually positive or eventually negative. As $(-x(n))_{n\geqslant -k}$ is also a solution of (E$_1$) or (E$_m$), we may restrict ourselves only to the case where $x(n) > 0$ for all large n. Let $n_1 \geqslant -k$ be an integer such that $x(n) > 0$ for all $n \geqslant n_1 \geqslant n_*$. Then, there exists $n_0 \geqslant n_1$ such that $x(\tau(n)) > 0,\ x(\sigma(n)) > 0,\ \forall n \geqslant n_0$.