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A Korovkin's type approximation theorem for periodic functions via the statistical summability of the generalized de la Vallée Poussin mean

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ABSTRACT

The main object of this paper is to prove a Korovkin type theorem for the test functions 1, $\cos x$, $\sin x$ in the space $C_{2\pi}(\mathbb{R})$ of all continuous 2π-periodic functions on the real line $\mathbb{R}$. Our analysis is based upon the statistical summability involving the idea of the generalized de la Vallée Poussin mean. We also investigate the rate of the de la Vallée Poussin statistical summability of positive linear operators in the space $C_{2\pi}(\mathbb{R})$. Finally, we provide an interesting illustrative example in support of our result.

1. Introduction, definitions and preliminaries

We shall denote by $\mathbb{N}$ the set of all natural numbers. Let $\mathbb{K} \subset \mathbb{N}$ and suppose that

$$\mathbb{K}_n := \{k : k \leqq n \quad \text{and} \quad k \in \mathbb{K}\}.$$

Then the natural density of $\mathbb{K}$ is defined by

$$d(\mathbb{K}) := \lim_{n\to\infty} \frac{|\mathbb{K}_n|}{n} = \lim_{n\to\infty} \frac{1}{n} |\{k : k \leqq n \quad \text{and} \quad k \in \mathbb{K}\}|,$$

if the limit exists. Here, and in other similar situations, by $|\Omega|$ we denote the number of elements in the enclosed set Ω. A given sequence $X = (x_n)$ is said to be statistically convergent to L if, for each $\epsilon > 0$, the set

$$\mathcal{K}_\epsilon := \{k : k \in \mathbb{N} \quad \text{and} \quad |x_k - L| \geqq \epsilon\},$$

has natural density zero (cf. [5,20]), that is, for each $\epsilon > 0$, we have

$$\lim_{n\to\infty} \frac{|\mathcal{K}_\epsilon|}{n} = \lim_{n\to\infty} \frac{1}{n} |\{k : k \leqq n \quad \text{and} \quad |x_k - L| \geqq \epsilon\}| = 0.$$

In this case, we write

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$L = \text{stat}\lim_{n\to\infty} x_n.$

We note here that every convergent sequence is statistically convergent, but not conversely.

Recently, Móricz [12] introduced the concept of statistical summability $(C,1)$ as follows.

Definition 1. Given a sequence $X = (x_n)$ for which

$$t_n := \frac{1}{n+1}\sum_{k=0}^{n} x_k, \tag{1.1}$$

we say that the sequence $X = (x_n)$ is statistically summable $(C,1)$ if

$\text{stat}\lim_{n\to\infty} t_n = L.$

In this case, we write

$L = (\text{C},1)\text{stat}\lim_{n\to\infty} x_n.$

The following idea was formerly given under the name of "λ-convergence" by Mursaleen and Noman [16]. We find it to be more suitable to call this notion as the "de la Vallée Poussin summability" in place of "λ-convergence."

Definition 2. Let $\Lambda = (\lambda_n)_{n=0}^{\infty}$ be a strictly increasing sequence of positive real numbers which tend to infinity as $n \to \infty$, that is, let

$$0 < \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots \quad \text{and} \quad \lim_{n\to\infty} \lambda_n = \infty.$$

Then, by defining the modified and generalized de la Vallée Poussin mean as follows:

$$V_n := \frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})x_k \quad (\lambda_{-1} := 0), \tag{1.2}$$

we say that the sequence $X = (x_n)$ is de la Vallée Poussin satistically summable to L as $n \to \infty$ if

$\text{stat}\lim_{n\to\infty} V_n = L,$

and we write

$(\text{VP})\text{stat}\lim_{n\to\infty} x_n = L.$

If we take $\lambda_n = n+1$ $(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})$, then the λ-convergence is reduced to the Cesàro summability or, equivalently, the $(C,1)$-summability and the de la Vallée Poussin statistical summability is reduced to the statistically $(C,1)$-summability. It is well known that every $(C,1)$-summable sequence is statistically $(C,1)$-summable, but not conversely.

Example 1. Consider the sequence $X = (x_n)$ defined by $x_n = (-1)^{n-1}$ for all $n \in \mathbb{N}$. Also let $\lambda_n = n+1$ $(n \in \mathbb{N}_0)$. Then the sequence X is λ-convergent to 0 and hence de la Vallée Poussin statistically convergent to 0, but it is neither convergent nor statistically convergent.

The following notations and conventions related to the function spaces will be used in our present investigation. By $F(\mathbb{R})$ we will denote the linear space of all real-valued functions defined in $\mathbb{R}$. And by $C(\mathbb{R})$ we will denote the space of all bounded and continuous functions defined in $\mathbb{R}$. The space $C(\mathbb{R})$ is equipped with the following norm:

$$\|f\|_{\infty} = \sup_{x\in\mathbb{R}} |f(x)| \quad \big(f \in C(\mathbb{R})\big).$$

The space of all 2π-periodic and continuous functions will be denoted by $C_{2\pi}(\mathbb{R})$, which is equipped with the supremum norm given by

$$\|f\|_{2\pi} = \sup_{x\in\mathbb{R}} |f(x)| \quad \big(f \in C_{2\pi}(\mathbb{R})\big).$$

In each of the above-mentioned cases, it is *tacitly* understood that the set $\mathbb{R}$ can indeed be replaced by any given interval I on the real axis.

We now recall the classical Korovkin first and second theorems as follows (see, for details, [6,7,1]). By definition, a given sequence $(\mathcal{T}_n)$ of linear operators is said to be *positive* if, for a given $x \in I \subset \mathbb{R}$, we have (see, for example, [18])

$$\mathcal{T}_n(f,x) \geqslant 0 \quad \big(f(x) \geqq 0;\ x \in I\big).$$