Applied Mathematics and Computation 228 (2014) 184–194

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Bifurcation analysis and chaos in a discrete reduced Lorenz system

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A R T I C L E I N F O

Keywords:
Discrete Lorenz system
Chaotic behavior
Neimark–Sacker bifurcation
Lyapunov exponents

A B S T R A C T

In this paper, the discrete reduced Lorenz system is considered. The dynamical behavior of the system is investigated. The existence and stability of the fixed points of this system are derived. The conditions for existence of a pitchfork bifurcation, flip bifurcation and Neimark–Sacker bifurcation are derived by using the center manifold theorem and bifurcation theory. The complex dynamics, bifurcations and chaos are displayed by numerical simulations.

1. Introduction

The discrete reduced Lorenz system is a discretisation of a pair of coupled differential equations which were used by Lorenz [1]. The discrete reduced Lorenz system is defined in the following way [1,2]:

$$\begin{cases} x_{n+1} = (1+ah)x_n - hx_ny_n, \\ y_{n+1} = (1-h)y_n + hx_n^2. \end{cases} \tag{1}$$

In [1,2] a numerical study for the system (1) was given. It it shown that it has a richer set of dynamical patterns than those observed in the continuous case, but a mathematical analysis is not given. In the current paper we present a theoretical analysis of the bifurcation phenomena. Jing and co-workers had applied the forward Euler method to a BVP oscillator, to the FitzHugh–Nagumo system, and to a predator–prey model [3–5]. After a qualitative analysis, and application of the center manifold theorem, they investigated the dynamical behavior of corresponding discrete systems. Bifurcations and chaos in an ecological model with delays were examined in [6]. Zhang et al. applied the forward Euler scheme to a simple predator–prey model [7]. They also studied the dynamical behavior using qualitative analysis and the center manifold theorem. Hopf bifurcation analysis in a delayed Nicholson blowflies equation was investigated in [8]. Elabbasy et al. had used the same techniques to study the dynamical behavior of Burgers mapping [9]. Also, Gao and Liu [10] studied the dynamical behaviors of a two-dimensional discrete system. Hu et al. [11] discussed the stability and bifurcations of a discrete predator–prey model with nonmonotonic functional response. Bifurcation and chaotic behavior in an epidemic model with nonlinear incidence rates were examined by Li et al. [12]. Christodoulou [13] introduced a discrete Hopf bifurcation for Runge–Kutta methods. Bischi and Tramontana [14] studied some local and global properties and bifurcations in discrete-time Lotka–Volterra models with an application to industrial clusters. Agiza et al. [15] investigated chaotic dynamics of a discrete prey–predator model with Holling Type II response and showed that the discrete system exhibits far richar dynamics compared to the

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http://dx.doi.org/10.1016/j.amc.2013.11.088

continuous model. Numerical simulations not only show consistency with the theoretical analysis but also exhibit the complex dynamical behavior. Jana [16] extended Agiza et al.'s [15] prey–predator with a Holling type II functional response incorporating prey refuge. Also, He provided the conditions of existence for stability, flip bifurcation, and Neimark–Sacker and found richer dynamics by using both theoretical and numerical analysis.

The discrete reduced Lorenz system introduced by Lorenz in [1]. He predicted chaotic dynamics when h is very large. The paper [1] is a milestone in the study of concept of computational chaos. Djellit et al. studied the fractal basins for Lorenz model in [17]. Also Djellit and Kara [18] provided numerical evidence of such a bifurcation for the appearance of invariant sets in this model. Recently, the information security using chaotic dynamics is a novel topic in the encryption research field. An encryption based on two-dimensional discretized chaotic maps were proposed [19,20]. The chaotic map based on encryption techniques have shown their superior performance, and it has been proved that in many aspects chaotic maps have important characteristics related to the fundamental requirements of conventional encryption algorithms [21]. For instance, aperiodicity (useful for one time pad cipher), sensitive dependence on initial conditions and system parameters (useful for confusion and diffusion processes), ergodicity and random-like behaviors (useful for producing output). So, a discrete reduced Lorenz system can be used in encryption.

In this study, the discrete reduced Lorenz system (1) is further investigated in details. Conditions will be derived for the existence of pitchfork bifurcation, flip bifurcation and Neimark–Sacker bifurcation by using the center manifold theorem and bifurcation theory [22,23].

This paper is organized as follows. In Section 2 the existence and stability of the fixed points of the map (1) is presented. The qualitative behavior and the bifurcations of this discrete Lorenz system are discussed, using qualitative theory and bifurcation theory. Also, conditions for the existence of a pitchfork bifurcation, a flip bifurcation, and a Neimark–Sacker bifurcation, are derived. Numerical simulation results are presented in Section 3 to verify the theoretical analysis and visualize the newly observed complex dynamics of the system. Finally, in Section 4 we present our conclusions.

2. Existence and stability of fixed points and bifurcations

In this section, we first determine the existence of the fixed points of the system (1), then investigate their stability by calculating the eigenvalues for the Jacobian matrix of the system (1) at each fixed point. Sufficient conditions for existence of pitchfork bifurcations, flip bifurcations and Neimark–Sacker bifurcations, are derived by using qualitative theory [22] and bifurcation theory [23].

By simple calculations the system (1) has the following three fixed points:

(i) $E_0(0,0)$ is trivial fixed point,
(ii) $E_1(\sqrt{a},a)$ is a non-trivial fixed point that exist for $a>0$, and
(iii) $E_2(-\sqrt{a},a)$ is another fixed point. Where E_1 and E_2 are symmetric fixed points around y-axis.

Next we will investigate the qualitative behavior of the system (1). The local dynamics of the system (1) in a neighborhood of a fixed point is dependent on the Jacobian matrix of (1). The Jacobian matrix of the system (1) at the state variable is given by

$$J(x,y)=\begin{pmatrix} 1+ah-hy & -hx \\ 2hx & 1-h \end{pmatrix}. \quad (2)$$

2.1. Bifurcation of $E_0(0;0)$

The following is Jacobian matrix at E_0:

$$J(E_0)=\begin{pmatrix} 1+ah & 0 \\ 0 & 1-h \end{pmatrix}, \quad (3)$$

has two eigenvalues $\lambda_1=1+ah$ and $\lambda_2=1-h$. Provided that $h\neq 0$, along line $a=0$ implies $\lambda_1=1$ and $\lambda_2=1-h$. For (a,h) crosses from $a<0$ to $a>0$, a bifurcation occurs. One can check that the condition for a pitchfork bifurcation is satisfied when $a=0$. As $4+2ah-2h-ah^2=0$, then $\lambda_1=-1$ and $\lambda_2=1-h$. For (a,h) crosses $4+2ah-2h-ah^2<0$ to $4+2ah-2h-ah^2>0$, a bifurcation occurs. One can check that the condition for a flip bifurcation (period doubling) is satisfied when $h=-\frac{2}{a}$.

If $a=0$, then $J(E_0)$ has two eigenvalues $\lambda_1=1$ and $\lambda_1=1-h$. If $h\neq 2$, then $|\lambda_2|\neq 1$, so the conditions of the occurrence of pitchfork bifurcation at E_0 is given by the following Lemma.

Lemma 1. *If $a=0$ and $h\neq 2$, the system (1) undergoes a pitchfork bifurcation at $E_0(0,0)$. Moreover, the system has only one fixed point.*