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Bifurcations and some new traveling wave solutions for the CH-γ equation

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ABSTRACT

In this paper, the CH-γ equation is investigated by employing the bifurcation theory and the method of phase portraits analysis. The dynamical behavior of equilibrium points and the bifurcations of phase portraits of the traveling wave system corresponding to this equation are discussed. Under some parameter conditions, some bounded traveling wave solutions such as solitary waves, peakons and periodic cusp waves are presented. Furthermore, based on the auxiliary equation, various new traveling wave solutions of parametric form are given. The previous results for this equation are extended.

1. Introduction

It is well known that the CH-γ (Camassa-Holm-γ) equation

$$m_t + c_0 u_x + u m_x + 2 m u_x = -\gamma u_{xxx}, \tag{1.1}$$

was proposed by Dullin et al. [1–3] as a model for the unidirectional nonlinear dispersive water wave with a surface tension, where $m = u - \alpha^2 u_{xx}$ is a momentum variable, the constants α^2 and γ/c_0 are squares of length scales and c_0 is the linear wave speed for the undisturbed water at rest at spatial infinity. This equation has been shown to be integrable [1] and asymptotically equivalent to other shallow water wave equations [2,3].

Eq. (1.1) can be rewritten as

$$u_t + c_0 u_x + 3 u u_x - \alpha^2 (u_{xxt} + u u_{xxx} + 2 u_x u_{xx}) = -\gamma u_{xxx}. \tag{1.2}$$

Guo and Liu [4] obtained some expressions of peaked wave solutions of Eq. (1.2). Tang and Yang [5] introduced an integral constant and extended the peaked wave solutions of Eq. (1.2). Deng [6] extended the peaked wave solutions of Eq. (1.2) by the first-integral method. In [7], Guo and Liu derived the compacton-like wave solutions and kink-like wave solutions of Eq. (1.2) for $\alpha^2 > 0$. For $\alpha^2 < 0$, Tang and Zhang [8] showed that Eq. (1.2) has periodic waves, compacton-like waves, kink-like waves and solitary waves. Chen et al. [9] solved Eq. (1.2) for analytical multiple soliton solutions with the Darboux transformation. Zhang [10] introduced a variable transformation such that Eq. (1.2) becomes the CH (Camassa-Holm) equation which has been studied successively by many authors (see [11–19] and the references therein). To determine how the dispersion parameters α and γ in Eq. (1.2) affect the isospectral content of its soliton solutions and the shape of its wave solutions, Bi [20] investigated the bifurcation of traveling wave solutions of Eq. (1.2) by employing qualitative analysis method. Unfortunately, the results in [20] are not complete for the sake of the special choice of the parameter c_0 in Eq. (1.2). In this

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paper, by using the bifurcation theory and the method of phase portraits analysis [21], we will discuss the bifurcations for all possible parameter conditions in detail. We will also try to give some new expressions of traveling wave solutions which contain parameter c_0, so these solutions possess more general form than those in Ref. [20].

Recently, based on the computer algebraic system like Maple and Mathematica, Many direct and effective methods [23–32] are proposed to construct new exact solutions of nonlinear partial differential equations. For example, Li et al. [23] applied *F*-expansion method to the sin-Gordon and sinh-Gordon equations. Zhang et al. [24] applied auxiliary method to mCH and mDP equations and obtained diversity of exact solutions. Ma and Lee [25] presented a transformed ration function method which puts many existing algebraic methods contained therein and applied it to the 3 + 1 dimensional Jimbo–Miwa equation. Rui et al. [26] proposed integral bifurcation method which includes the bifurcation method and auxiliary equation method and applied it to the higher order wave equation of KdV type (III). We are inspired by the above-mentioned algebraic methods and note the singularity of the planar dynamical system corresponding to Eq. (1.2). Therefore we will search for some traveling wave solutions of parametric form by using auxiliary equation.

The rest of this paper is organized as follows. In Section 2, we discuss the dynamical behavior of equilibrium points and the bifurcations of phase portraits of corresponding traveling wave system. In Section 3, we obtain three types of bounded traveling wave solutions, which include solitary wave solutions, peakons and periodic cusp wave solutions. Meanwhile, we show that the limits of smooth solitary wave solutions and periodic cusp wave solutions are peakons. In Section 4, some traveling wave solutions of parametric form are given. A short conclusion is given in Section 5.

2. Bifurcation analysis

We first make the traveling wave transformation as follows:

$$u(x,t) = \phi(\xi), \quad \xi = x - ct, \tag{2.1}$$

where $c \neq 0$ is the wave speed. Then Eq. (1.2) reduces to be

$$-c\phi' + c_0\phi' + 3\phi\phi' - \alpha^2(\phi\phi''' + 2\phi'\phi'' - c\phi''') = -\gamma\phi'''. \tag{2.2}$$

Integrating (2.2) with regard to ξ once and setting the constant of integration to zero, we have

$$(\alpha^2\phi - c\alpha^2 - \gamma)\phi'' = \frac{3}{2}\phi^2 + (c_0 - c)\phi - \frac{\alpha^2}{2}(\phi')^2, \tag{2.3}$$

Letting $y = \phi'$, then we obtain the following planar dynamical system

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{\frac{3}{2}\phi^2 + (c_0 - c)\phi - \frac{\alpha^2}{2}y^2}{\alpha^2\phi - c\alpha^2 - \gamma}, \tag{2.4}$$

which is called traveling wave system. This is a planar Hamiltonian system with Hamiltonian function

$$H(\phi, y) = (\alpha^2\phi - c\alpha^2 - \gamma)y^2 - \phi^3 + (c - c_0)\phi^2 = h, \tag{2.5}$$

where h is a constant.

Note that (2.4) has a singular line $\phi = c + \frac{\gamma}{\alpha^2}$. To avoid the line temporarily, we make the transformation $d\xi = (\alpha^2\phi - c\alpha^2 - \gamma)d\tau$. Under this transformation, Eq. (2.4) becomes

$$\frac{d\phi}{d\tau} = (\alpha^2\phi - c\alpha^2 - \gamma)y, \quad \frac{dy}{d\tau} = \frac{3}{2}\phi^2 + (c_0 - c)\phi - \frac{\alpha^2}{2}y^2. \tag{2.6}$$

System (2.4) and (2.6) have the first integral as (2.5). Consequently, system (2.6) has the same topological phase portrait as system (2.4) except for the singular line $\phi = c + \frac{\gamma}{\alpha^2}$. For the new system (2.6), $\phi = c + \frac{\gamma}{\alpha^2}$ is an invariant line solution.

For a fixed h, (2.5) determines a set of invariant curves of system (2.6). As h is varied, (2.5) determines different families of orbits of system (2.6) having different dynamical behaviors. Let $M(\phi_e, y_e)$ be the coefficient matrix of the linearized version of system (2.6) at the equilibrium point (ϕ_e, y_e), then

$$M(\phi_e, y_e) = \begin{pmatrix} \alpha^2 y_e & \alpha^2\phi_e - c\alpha^2 - \gamma \\ 3\phi_e + c_0 - c & -\alpha^2 y_e \end{pmatrix} \tag{2.7}$$

and at this equilibrium point, we have

$$J(\phi_e, y_e) = \det M(\phi_e, y_e) = -\alpha^4 y_e^2 - (\alpha^2\phi_e - c\alpha^2 - \gamma)(3\phi_e + c_0 - c), \tag{2.8}$$

$$p(\phi_e, y_e) = \operatorname{trace} M(\phi_e, y_e) = 0. \tag{2.9}$$

By the theory of planar dynamical systems, for an equilibrium point of an integrable system, if $J < 0$, then this equilibrium point is a saddle point; it is a center point if $J > 0$ and $p = 0$; if $J = 0$ and the Poincaré index of the equilibrium point is 0, then it is a cusp.