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Convergence and stability of the semi-tamed Euler scheme for stochastic differential equations with non-Lipschitz continuous coefficients

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ABSTRACT

Recently, explicit tamed schemes were proposed to approximate the SDEs with the non-Lipschitz continuous coefficients. This work proposes a semi-tamed Euler scheme, which is also explicit, to solve the SDEs with the drift coefficient equipped with the Lipschitz continuous part and non-Lipschitz continuous part. It is shown that the semi-tamed Euler converges strongly with the standard order one-half to the exact solution of the SDE. We also investigate the stability inheritance of the semi-tamed Euler schemes and reveal that this scheme does have advantage in reproducing the exponential mean square stability of the exact solution. Numerical experiments confirm the theoretical analysis.

1. Introduction

In this work, we study the numerical solution to the stochastic differential equations (SDEs) of the form

$$dx_t = \mu(x_t)dt + \sigma(x_t)dW_t, \qquad x(0) = \xi, \tag{1.1}$$

where W_t is a m-dimensional Brownian motion, $\sigma : \mathbb{R}^d \mapsto \mathbb{R}^{d\times m}$ is global Lipschitz continuous, $\mu(x) = f(x) + g(x)$ is globally one-side Lipschitz continuous, the function $f : \mathbb{R}^d \mapsto \mathbb{R}^d$ is the Lipschitz continuous part of μ, and $g : \mathbb{R}^d \mapsto \mathbb{R}^d$ is non-Lipschitz continuous part. The SDE of this form does involve many stochastic models in the real world, such as stochastic Duffing–van der Pol oscillator [1,2], stochastic Lorenz equation [3,2], experimental psychology model [2], stochastic Ginzburg–Landau equation [4,2], stochastic Lotka–Volterra equations [5,4,2] and volatility processes [4,2], to name a few. Numerical analysis is an important tool in studying stochastic models, since most SDEs cannot be solved explicitly. In judging the quality of a numerical scheme, it is necessary to examine its convergence and stability.

Since the drift coefficient is non-Lipschitz continuous, the classic explicit Euler–Maruyama (EM) method, investigated in Kloeden and Platen [4], Maruyama [6] and Milstein [7] for approximating the SDEs with globally Lipschitz continuous coefficients, may not converge in the strong mean square sense to the exact solution. Hutzenthaler et al. [8,9] shown that absolute moments of the explicit EM approximation for a SDE with a superlinearly growing and globally one-sided Lipschitz

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continuous drift coefficient, diverge to infinity at a finite time point. Although the implicit method, such as the backward Euler, split-step backward Euler methods, holds the strong convergence [10,11], it requires additional computation effort to solve a implicit system. Recently, Hutzenthaler et al. [12] proposed a new explicit approximation, called drift-tamed Euler method

$$Y_{n+1}^N = Y_n^N + \frac{\mu(Y_n^N)T/N}{1+\|\mu(Y_n^N)\|T/N} + \sigma(Y_n^N)\Delta W_n^N, \tag{1.2}$$

where $\Delta W_n^N = W_{(n+1)T/N} - W_{nT/N}$. It is shown that if the drift coefficient function is globally one-side Lipschitz continuous and has an at most polynomially growing derivative, the drift-tamed Euler method (1.2) converge strongly to the exact solution with the standard convergence order 0.5. Following the ideas, Wang and Gan [13] proposed the tamed Milstein method and proved it converge strongly to the exact solution with the convergence order 1. For the SDEs with non-Lipschitz coefficients, Hutzenthaler et al. [2] also proposed a series of the numerical approximations, such as partially drift-implicit approximation, linear implicit scheme, drift-truncated Euler and increment-tamed Euler, etc. Sauer and Stannat [14] proved strong convergence of the finite differences approximation in space for stochastic reaction diffusion equations with multiplicative noise under a one-sided Lipschitz condition using the tamed idea.

Numerical stability for SDEs is primarily concerned with ascertaining for what values of stepsize does a particular numerical method replicate the stability properties of the exact solution. Under the globally Lipschitz condition, the explicit Euler–Maruyama scheme can reproduce the exponential mean square stability of the exact solution (see [15–19]), but for the SDEs (1.1) with non-Lipschitz coefficients, Higham et al. [20] gave a counterexample to show that the explicit Euler scheme may be explosive even though the exact solution is stable. Although the implicit scheme can share the stability of the exact solution (see [17,15,21]), it also requires additional computation effort to solve a implicit system. Therefore, it is necessary to seek for a new numerical scheme, which is explicit and can replicate the stability properties of the exact solution for the SDEs with non-Lipschitz continuous coefficients.

Inspired by the literature [22,12,2], we derive a non-Lipschitz term tamed approximation for SDE (1.1):

$$Y_{n+1}^N = Y_n^N + f(Y_n^N)T/N + \frac{g(Y_n^N)T/N}{1+\|g(Y_n^N)\|T/N} + \sigma(Y_n^N)\Delta W_n^N. \tag{1.3}$$

We refer to this approximation as semi-tamed Euler scheme. The increment function defined in [2] is $\phi(x,h,y) = f(x)h + \frac{g(x)}{1+\|g(x)\|h}h + \sigma(x)y$. We will firstly show that the semi-tamed Euler scheme converges strongly to the exact solution with the standard convergence order 0.5. Then we prove that the semi-tamed Euler scheme (1.3) can not only reproduce the exponential mean square stability of the exact solution, but also work better than the tamed Euler (1.2), backward Euler and split-step backward Euler schemes at stability preservation under a stepsize restriction.

The rest of the paper is organized as follows. Section 2 begins with notations and estimation of the moment. Section 3 aims to show the strong convergence of the semi-tamed Euler scheme. Section 4 devotes to investigating the exponential mean square stability of the semi-tamed scheme for SDEs. Section 6 provides some numerical experiments for demonstration.

2. Estimation of the *p*th moments

Throughout the whole article, unless otherwise specified, we use the following notations. Let $T \in (0,\infty)$ be a fixed real number and $N \in \mathbb{N}$ be the step number of the uniform mesh with the stepsize T/N. Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathfrak{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions and W_t be a m-dimensional Brownian motion defined on this probability space. Moreover, we use the notation $\|x\| = (|x_1|^2 + \cdots + |x_k|^2)^{1/2}$, $\langle x, y\rangle = x_1y_1 + x_2y_2 + \cdots + x_ky_k$ for all $x, y \in \mathbb{R}^k$, $k \in \mathbb{N}$, and $\|A\| := \sup_{x\in\mathbb{R}^l,\ \|x\|\leqslant 1}\|Ax\|$ for all $A \in \mathbb{R}^{k\times l}$, $k, l \in \mathbb{N}$. $a \vee b$ represents $\max\{a,b\}$ and $a \wedge b$ denotes $\min\{a,b\}$. In this work, we also make the following assumptions.

Assumption 2.1. Let $g(x)$ be a continuously differentiable function and there exist positive constants $K > 1$ and $c > 0$, such that for any $x, y \in \mathbb{R}^d$

$$\|f(x) - f(y)\| \vee \|\sigma(x) - \sigma(y)\| \leqslant K\|x-y\|, \tag{2.1}$$

$$\langle x-y, \mu(x) - \mu(y)\rangle \leqslant K\|x-y\|^2, \tag{2.2}$$

$$\|g'(x)\| \leqslant K\|x\|^c. \tag{2.3}$$

Remark 2.1. There many stochastic models hold Assumption 2.1, such as stochastic Ginzburg–Landau equation, stochastic Verhulst equation, volatility processes (see [2]). Note that conditions (2.1) and (2.2) imply that SDE (1.1) admits the *p*th moment bounded solution in any finite time *T* (see [23]).