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Nonlinear fractional differential equations with integral boundary value conditions

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ABSTRACT

In this paper, we are interested in the study of the existence of the solutions of a class of nonlinear boundary value problem of fractional differential equations with integral boundary conditions. We make an exhaustive study of the sign of the related Green's function and obtain the exact values for which it is positive on the whole square of definition. The existence of solutions follows from the definition of suitable cones on Banach spaces.

1. Introduction

There is a huge number of phenomena that are modeled by different equations involving fractional order derivatives. On the monographs [1–3] the reader can found examples that appear in different fields as physics, chemistry, aerodynamics or electrodynamics. The main difference with the usual derivative is that the definition of the fractional one involves all the values of the function up to the point t in which it is valued. This fact makes it more useful for problems with some kind of memory, [4]. As it can be seen in recent published articles, the study of this subject is growing up in the last years, see [5–19] and the references therein. We make special mention to the recent papers [20,21] in which, among other results, it has been proved the existence and uniqueness of solution of different types of nonlinear fractional Cauchy problems. Furthermore, in [22] is deduced the existence of solutions for impulsive fractional stochastic differential equations with infinite delay. In these three papers the fixed point theory plays a fundamental role in the proofs of the obtained results.

This paper considers a kind of integral boundary conditions. As it has been stated in [23], this type of conditions appear in different real phenomena as, among others, blood flow problems, chemical engineering, thermo-elasticity or population dynamics, see [24–33] and the references therein.

To be concise, in this paper we are concerned with the study of the existence of solutions of the following nonlinear fractional differential equations with integral boundary value conditions

$$D^{\alpha}u(t)+f(t,u(t))=0,\quad 0<t<1, \tag{1}$$

$$u(0)=u'(0)=0,\quad u(1)=\lambda\int_0^1 u(s)ds. \tag{2}$$

where $2<\alpha\leqslant 3,\ 0<\lambda,\ \lambda\neq\alpha,\ D^{\alpha}$ is the Rieman–Liouville fractional derivative and f is a continuous function.

In a first moment we obtain the exact expression of the Green's function related to the linear problem

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$$D^{\alpha}u(t) + y(t) = 0, \quad 0 < t < 1, \tag{3}$$

coupled with the integral boundary conditions (2).

Once we have obtained such expression, we study the values of λ for which the Green's function is positive on $(0,1)\times(0,1)$ and, moreover we deduce some suitable properties that relates the expression of $G(t,s)$ and $G(1,s)$. This properties will be fundamental to construct a suitable cone in the spaces of the continuous functions and so, under additional conditions on the behavior of function f at 0 and at ∞, we deduce the existence of positive solutions of problem (1) and (2).

In some sense, the results given in this work follow similar steps to the ones obtained in [23] for the problem

$$\begin{cases} {}^{C}D^{\alpha}u(t) + f(t,u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u''|(0) = 0, \quad u(1) = \lambda \int_0^1 u(s)ds, \end{cases}$$

where $2 < \alpha < 3,\ 0 < \lambda < 2,\ {}^{C}D^{\alpha}$ is the Caputo fractional derivative and $f : [0,1]\times[0,\infty) \to [0,\infty)$ is a continuous function.

2. Study of the Green's function

In this section, we obtain the exact expression of the Green's function related to the linear fractional differential equation with integral boundary value conditions (3) and (2). The result is the following.

Theorem 2.1. *Let $2 < \alpha \leqslant 3$ and $\lambda \neq \alpha$. Assume $y \in C[0,1]$, then the problem (3) and (2) has a unique solution $u \in C^1[0,1]$, given by the expression*

$$u(t) = \int_0^1 G(t,s)y(s)ds,$$

where

$$G(t,s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s)-(\alpha-\lambda)(t-s)^{\alpha-1}}{(\alpha-\lambda)\Gamma(\alpha)}, & 0 \leqslant s \leqslant t \leq 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s)}{(\alpha-\lambda)\Gamma(\alpha)}, & 0 \leqslant t \leqslant s \leq 1. \end{cases} \tag{4}$$

Proof. It is very well known, see [1, Theorem 3.1] or [9, Lemma 2.2], that Eq. (3) is equivalent to the following integral equation

$$u(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3}.$$

Since $u(0) = u'(0) = 0$, we deduce that

$$u(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + c_1 t^{\alpha-1}.$$

Finally, condition $u(1) = \lambda \int_0^1 u(s)ds$ implies that

$$c_1 = \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + \lambda \int_0^1 u(s)ds.$$

Hence, we have the following form

$$u(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + t^{\alpha-1}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + \lambda t^{\alpha-1}\int_0^1 u(s)ds. \tag{5}$$

Let $\int_0^1 u(s)ds = A$, then, from the previous equality, we deduce that

$$\begin{aligned} A = \int_0^1 u(t)dt &= -\int_0^1\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds\,dt + \int_0^1\int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds\,dt + \lambda A\int_0^1 t^{\alpha-1}dt \\ &= -\int_0^1 \frac{(1-s)^{\alpha}}{\alpha\Gamma(\alpha)} y(s)ds + \frac{1}{\alpha}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + \frac{\lambda}{\alpha}A. \end{aligned} \tag{6}$$

So, expression (2) implies that

$$A = -\frac{1}{\alpha-\lambda}\int_0^1 \frac{(1-s)^{\alpha}}{\Gamma(\alpha)} y(s)ds + \frac{1}{\alpha-\lambda}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds.$$

Replacing this value in (2), we arrive at the following expression for function u: