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Stochastic linear quadratic optimal control with constraint for discrete-time systems ☆

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ABSTRACT

In this paper, we consider linear quadratic optimal control with constraint for discrete-time stochastic systems with state and disturbance dependent noise. With the aid of the Lagrange multiplier theorem, we present a necessary condition under which the problem is well posed and a state feedback solution can be derived. Moreover, a sufficient condition is introduced for the case in which the quadratic-term matrices are non-negative. In a way, the previous results on stochastic linear quadratic optimal control without constraint can be regarded as corollaries of the theorems of this paper.

1. Introduction

The linear quadratic (LQ) optimal control problem was pioneered by Kalman [1], which has been playing a central role in modern control theory. In recent decades, the definite LQ control problem has been investigated extensively by many researchers [2,3]. Stochastic LQ control problem for the Itô systems was initiated by Wonham [4], while the nonlinear regulator problem was discussed in [5]. Some of the works on this subject revealed that for stochastic Itô systems, even if the state and control weighting matrices Q and R are indefinite, the corresponding stochastic LQ problem may be still well posed, which was first found in [6], and has inspired a series of works [7–9]. For the discrete-time LQ control problems with control and or state dependent noises, there have been some works in literature. One early work [10] deals with a special case, whose systems are described by a difference equation in which both the system matrix and control matrix are multiplied by white, possibly correlated, scalar random sequences. In another article [11] the optimal control law is derived for the systems with only control dependent noises. It is worth noting that the state weight matrix is nonnegative and the control weight matrix is positive definite in both papers.

For discrete-time LQ problem, the control weighting matrix is not required to be positive definite, even negative when uncertainty factors are involved in the system [12]. Most previous researchers mainly studied indefinite stochastic LQ problems without constraints. However, some constraints are considerable importance in many physical systems. The finite time indefinite stochastic LQ control with linear terminal state constraint was discussed [13]. It is a valuable research topic to generalize those results to the discrete-time systems.

In this paper, we concentrate our attention on the finite horizon discrete-time indefinite stochastic LQ control with linear terminal constraint. It will be shown that the existence of optimal linear state feedback control by means of Lagrange multiplier theorem. The outline of this paper is organized as follows. In Section 2, we give some definitions and preliminaries. Section 3 contains our main theorems of discrete-time LQ control. A necessary condition and a sufficient

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condition for the existence of optimal linear state feedback control are respectively derived. Finally, Section 4 concludes the paper.

For convenience, we make use of the following basic notation in this paper: A' is the transpose of a matrix A; $tr(A)$ the trace of a square matrix A; $A > 0 (A \geqslant 0)$ means that A is positive definite (positive semi-definite) symmetric matrix; E represents the mathematical expectation; R^k the k-dimensional real vector space with the usual inner product $\langle \cdot, \cdot \rangle$and the corresponding 2-norm $\|\cdot\|$; $R^{m\times n}$ the vector space of all $m \times n$ matrices with entries in R; $S_n(R)$ the set of all real symmetric matrices; $N_t = \{0, 1, 2, \ldots, t\}$.

2. Definitions and preliminaries

Consider the following discrete-time stochastic system:

$$x(t+1) = [A_0(t)x(t) + B_0(t)u(t)] + [C_0(t)x(t) + D_0(t)u(t)]\omega(t), \quad t \in N_{T-1}, \tag{1}$$

$$b_{i1}x_1(T) + b_{i2}x_2(T) + \cdots + b_{in}x_n(T) = \xi_i \quad (i = 1, 2, \ldots, r). \tag{2}$$

where $x(0) = x_0 \in R^n, x(t) \in R^n$ and $u(t) \in R^m$ are respectively the system state and controlled input. $A_0(t), C_0(t) \in R^n, B_0(t), D_0(t) \in R^{n\times m}, t \in N_{T-1}$ are matrix-valued functions with appropriate dimensions. $\omega(t) \in R$ is a sequence of real random variables defined on a complete probability space $\{\Omega, F, \mu\}$, which is a wide sense stationary, second-order process with $E(\omega(t)) = 0$ and $E(\omega(t)\omega(s)) = \delta_{st}$ with δ_{st} being a Kronecker function. We denote F_t the σ-algebra generated by $\omega(s)$, i.e., $F_t = \sigma(\omega(s) : s \in N_t)$. $u(\cdot)$ belongs to the admissible control set $U_{ad} = \left\{u(t) \in R^m : E\sum_{t=0}^{T}|u(t)|^2 < +\infty\right\}$. ξ_i is F_T measurable square integrable stochastic process, namely $E|\xi_i| < +\infty$. b_{ij} is given real constant, $i = 1, 2, \ldots, r; j = 1, 2, \ldots n$. Let $N_{r\times n} = (b_{ij})_{r\times n}, \xi = (\xi_1, \xi_2, \ldots, \xi_r)'$, then (2) can be rewritten as $Nx(T) = \xi$, where suppose N has row full rank.

We first give some useful definitions and lemmas that are necessary for the proofs of our main results.

Definition 2.1 [14]. Let X be a vector space, Y a normed space, and T a transformation defined on a domain $D \subset X$ and having range $R \subset Y$. Let $x \in D$ and let h be arbitrary in X. If the limit

$$\delta T(x; h) = \lim_{\alpha\to 0}\frac{1}{\alpha}[T(x + \alpha h) - T(x)], \tag{3}$$

exists, it is called the Gateaux differential of T at x with increment h. If the limit (3) exists for each $h \in X$, the transformation T is said to be Gateaux differentiable at x.

Definition 2.2 [14]. Let T be a transformation defined on an open domain D in a normed space X and having range in a normed space Y. If for fixed $x \in D$ and each $h \in X$ there exists $\delta T(x; h) \in Y$ which is linear and continuous with respect to h such that

$$\lim_{\|h\|\to 0}\frac{\|T(x + h) - T(x) - \delta T(x; h)\|}{\|h\|} = 0. \tag{4}$$

Then T is said to be Frechet differentiable at x and $\delta T(x; h)$ is said to be the Frechet differential of T at x with increment h.

Definition 2.3 [14]. Let T be a continuously Frechet differentiable transformation from an open set D in a Banach space X into a Banach space Y. If $x_0 \in D$ is such that $\delta T(x_0; h)$ maps X onto Y, the point x_0 is said to be a regular point of the transformation T.

Lemma 2.4 (*Lagrange multiplier [14]*). *If the continuously Frechet differentiable functional f has a local extremum under the constant $H(x) = 0$ at the regular point x_0, then there exists an element $z_0^* \in Z^*$ such that the Lagrangian functional*

$$L(x) = f(x) + z_0^* H(x) \tag{5}$$

is stationary at x_0, i.e. $f'(x_0) + z_0^* H'(x_0) = 0$.

For later use, we recall the pseudo-inverse of a matrix.

Lemma 2.5 [15]. *Let a matrix $M \in R^{m\times n}$ be given. Then there exists a unique matrix $M^+ \in R^{n\times m}$, which is called the Moore–Penrose pseudo inverse of M, such that*

$$\begin{cases} MM^+M = M, \; M^+MM^+ = M^+, \\ (MM^+)' = MM^+, \; (M^+M)' = M^+M. \end{cases}$$