Applied Mathematics and Computation 228 (2014) 320–328

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Part-metric and its applications in discrete systems

Wanping Liu [a,b,*], Xiaofan Yang [a], Xinzhi Liu [b], Stevo Stević [c]

[a] College of Computer Science, Chongqing University, Chongqing 400044, China
[b] Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada
[c] Mathematical Institute of the Serbian Academy of Sciences, Knez Mihailova 36/III, 11000 Beograd, Serbia

ARTICLE INFO

Keywords:
Part-metric
Global asymptotic stability
Difference equation
Cyclic discrete dynamic system

ABSTRACT

This paper applies the *part-metric* method to study some types of higher-order symmetric difference equations with several different exponential parameters. These difference equations are proved to have unique equilibria and some useful inequalities regarding the difference equation functions are formulated. By use of the part-metric and a result given by Kruse and Nesemann (1999) [8], some sufficient conditions on the parameters are given to guarantee the global asymptotic stability of the equilibria. Furthermore, by the part-metric defined on matrices, this approach is also applicable to show the global asymptotic stability of some cyclic discrete dynamic systems. The results of this paper are considered a big improvement over many existing results found in the literature.

1. Introduction

The investigation of difference equations and discrete dynamic systems, which usually describe the evolution of certain phenomena over the course of time, undergoes a very long history. Although some difference equations and systems can be solved (see, e.g. [24–26] and the references therein), it is known that we cannot explicitly solve all them. In recent years, there has been an increasing interest in studying discrete dynamic systems which are not derived from differential equations. Especially, studying properties of rational difference equations has been an area of intense interest. In the literature, the *semi-cycle analysis*, *transformation method*, *part-metric method* are three main approaches often used to investigate the behavior of solutions to some type of symmetric difference equations and discrete dynamic systems. The semi-cycle analysis method is very effective in studying difference equations with lower orders and become extremely hard to deal with equations with higher orders.

In [4] did the authors employ a *transformation*, a better method than the *semi-cycle analysis* due to its effectiveness in dealing with higher-order equations and systems, to prove that all positive solutions to the difference equation

$$y_n = \frac{y_{n-k} + y_{n-l}}{1 + y_{n-k}y_{n-l}}, \quad n \in \mathbb{N}_0, \tag{1}$$

where $1 \leqslant k < l$, converge to the unique positive equilibrium $\bar{y} = 1$, while in [5] they improved this result to the difference equation

$$y_n = \frac{y_{n-k} + y_{n-l} + y_{n-m} + y_{n-k}y_{n-l}y_{n-m}}{1 + y_{n-k}y_{n-l} + y_{n-k}y_{n-m} + y_{n-m}y_{n-l}}, \quad n \in \mathbb{N}_0, \tag{2}$$

* Corresponding author at: College of Computer Science, Chongqing University, Chongqing 400044, China.
E-mail address: lwphe@163.com (W. Liu).

0096-3003/$ - see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.amc.2013.11.073

where $1 \leqslant k < l < m$, avoiding *semi-cycle analysis* often appearing in the literature. Note that, in 2007, Li applied the *semi-cycle analysis* method, which is usually effective for lower-order difference equations and is not essential in studying equations of arbitrary orders, to study a special case of Eq. (2) with $k = 1,\ l = 2,\ m = 4$ in [9] and another particular case $k = 2,\ l = 3,\ m = 4$ in [10], respectively. Furthermore, in [2] Berg and Stević shown that the *semi-cycle analysis* is unnecessarily complicated and they also presented several simple methods which can describe it in more elegant ways, as well as in more general settings.

By modifying the methods and ideas appearing in [4,5], the authors of [14] proved that positive solutions to the following generalized difference equations

$$y_n = \frac{y^r_{n-k} + y^r_{n-l}}{1 + y^r_{n-k}y^r_{n-l}}, \quad n \in \mathbb{N}_0 \tag{3}$$

and

$$y_n = \frac{y^r_{n-k}y^r_{n-l}y^r_{n-m} + y^r_{n-k} + y^r_{n-l} + y^r_{n-m}}{y^r_{n-k}y^r_{n-l} + y^r_{n-k}y^r_{n-m} + y^r_{n-l}y^r_{n-m} + 1}, \quad n \in \mathbb{N}_0, \tag{4}$$

where $1 \leqslant k < l < m$ and $r \in (0, 1]$, converge to the unique equilibrium.

Definition 1.1. Define the following symmetric functions, see equations (6)–(9) in [23]:

$$P^{2m}_{2m-1}(x_1, x_2, \ldots, x_{2m}) = \sum_{\substack{i=1\\ i\ \mathrm{odd}}}^{2m-1} \sum_{\substack{\{t_1,t_2,\ldots,t_i\}\subset S_{2m}\\ t_1<t_2<\cdots<t_i}} x_{t_1}x_{t_2}\ldots x_{t_i}, \tag{5}$$

$$P^{2m}_{2m}(x_1, x_2, \ldots, x_{2m}) = 1 + \sum_{\substack{i=2\\ i\ \mathrm{even}}}^{2m} \sum_{\substack{\{t_1,t_2,\ldots,t_i\}\subset S_{2m}\\ t_1<t_2<\cdots<t_i}} x_{t_1}x_{t_2}\ldots x_{t_i}, \tag{6}$$

$$P^{2m+1}_{2m+1}(x_1, x_2, \ldots, x_{2m+1}) = \sum_{\substack{i=1\\ i\ \mathrm{odd}}}^{2m+1} \sum_{\substack{\{t_1,t_2,\ldots,t_i\}\subset S_{2m+1}\\ t_1<t_2<\cdots<t_i}} x_{t_1}x_{t_2}\ldots x_{t_i}, \tag{7}$$

$$P^{2m+1}_{2m}(x_1, x_2, \ldots, x_{2m+1}) = 1 + \sum_{\substack{i=2\\ i\ \mathrm{even}}}^{2m} \sum_{\substack{\{t_1,t_2,\ldots,t_i\}\subset S_{2m+1}\\ t_1<t_2<\cdots<t_i}} x_{t_1}x_{t_2}\ldots x_{t_i}, \tag{8}$$

where $m \in \mathbb{N}, S_{2m} = \{1, 2, \ldots, 2m\}$, and $S_{2m+1} = \{1, 2, \ldots, 2m + 1\}$.

By the *transformation* method, Stević [23] confirmed the Conjecture 4.1 posed in [14] by proving that positive solutions to the following difference equations

$$y_n = \frac{P^{2m}_{2m-1}\left(y^r_{n-k_1}, y^r_{n-k_2}, \ldots, y^r_{n-k_{2m}}\right)}{P^{2m}_{2m}\left(y^r_{n-k_1}, y^r_{n-k_2}, \ldots, y^r_{n-k_{2m}}\right)}, \quad n \in \mathbb{N}_0, \tag{9}$$

and

$$y_n = \frac{P^{2m+1}_{2m+1}\left(y^r_{n-k_1}, y^r_{n-k_2}, \ldots, y^r_{n-k_{2m+1}}\right)}{P^{2m+1}_{2m}\left(y^r_{n-k_1}, y^r_{n-k_2}, \ldots, y^r_{n-k_{2m+1}}\right)}, \quad n \in \mathbb{N}_0, \tag{10}$$

where $r \in (0, 1], m \in \mathbb{N}, 1 \leqslant k_1 < k_2 < \cdots < k_{2m} < k_{2m+1}$, converge to the unique positive equilibrium $\bar{y} = 1$. Shortly after this work, the authors of [11] used a new approach called "frame sequences" method, which is inspired by [15], to give another different proof to similar results to Eq. (10) and the following difference equation

$$y_n = \frac{P^{2m+1}_{2m}\left(y^r_{n-k_1}, y^r_{n-k_2}, \ldots, y^r_{n-k_{2m+1}}\right)}{P^{2m+1}_{2m+1}\left(y^r_{n-k_1}, y^r_{n-k_2}, \ldots, y^r_{n-k_{2m+1}}\right)}, \quad n \in \mathbb{N}_0, \tag{11}$$

where $r \in (0, 1], m \in \mathbb{N}, 1 \leqslant k_1 < k_2 < \cdots < k_{2m} < k_{2m+1}$, which is the inverse of Eq. (10).

Remark 1.1. Note that for $m = 1$ we have that $P^2_1(x_1, x_2) = x_1 + x_2,\ P^2_2(x_1, x_2) = x_1x_2 + 1,\ P^3_3(x_1, x_2, x_3) = x_1x_2x_3 + x_1 + x_2 + x_3$ and $P^3_2(x_1, x_2, x_3) = x_1x_2 + x_1x_3 + x_2x_3 + 1$, thus in this case Eqs. (9) and (10) are reduced to Eqs. (3) and (4),respectively.

This paper aims to adapt *part-metric* to investigate difference equations of the form:

$$x_n = f_p(x_{n-k_1}, x_{n-k_2}, \ldots, x_{n-k_p}), \quad n \in \mathbb{N}_0, \tag{12}$$

$$x_n = \widetilde{f}_p(x_{n-k_1}, x_{n-k_2}, \ldots, x_{n-k_p}), \quad n \in \mathbb{N}_0, \tag{13}$$