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# On the number of zeros of Abelian integrals for a kind of quartic Hamiltonians ☆

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ABSTRACT

An explicit upper bound $B(n)$ is derived for the number of zeros of Abelian integrals $I(h) = \oint_{\Gamma_h} g(x,y)\,dx - f(x,y)\,dy$ on the open interval $(0, 1/4)$, where $\Gamma_h$ is an oval lying on the algebraic curve $H(x,y) = x^2 + y^2 - x^4 + ax^2y^2 + y^4$ with $a > -2$, $f(x,y)$ and $g(x,y)$ are polynomials in $x$ and $y$ of degrees not exceeding $n$. Assume $I(h)$ not vanish identically, then $B(n) \le 3\left[\frac{n-1}{4}\right] + 12\left[\frac{n-3}{4}\right] + 23$.



## 1. Introduction and main results

Consider the perturbated polynomial Hamiltonian systems:

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial y} + \varepsilon f(x,y), \\ \dot{y} = -\frac{\partial H}{\partial x} + \varepsilon g(x,y), \end{cases} \qquad ((\mathrm{H})_\varepsilon)$$

where $\varepsilon$ is a small parameter, $H(x,y), f(x,y)$ and $g(x,y)$ are real polynomials in $x$ and $y$ with $\deg H = d+1, \max\{\deg f, \deg g\} = n$. $H(x,y)$ is called a first integral of the vector field $(H)_{\varepsilon=0}$. Suppose that the Hamiltonian system $(H)_{\varepsilon=0}$ has at least one center. Consider the continuous family of ovals

$$\Gamma_h \subset \{(x,y) \in \mathbb{R}^2 : H(x,y) = h\},$$

which are defined on a maximal open interval $\Sigma = (h_c, h_s)$. Then the zeros of the displacement function

$$P_\varepsilon(h) - h = \varepsilon I(h) + O(\varepsilon^2),$$

gives the limit cycles of $(H)_\varepsilon$. The first coefficient $I(h)$ in the above expansion in $\varepsilon$ is given by

$$I(h) = \oint_{\Gamma_h} g(x,y)\,dx - f(x,y)\,dy, \quad h \in \Sigma, \tag{1}$$

which is called the Abelian integral. Then the weakened Hilbert 16th problem, initially posed by Arnold[1], is to find an upper bound $Z(d,n)$ of the number of zeros of the Abelian integral. Khovansky[7] and Varchenko[13] proved independently that it depends only on the degrees of polynomials $H(x,y), f(x,y)$ and $g(x,y)$, but an explicit expression for $Z(d,n)$ is unknown.

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It is natural to think about the possibility to find $Z(d,n)$ exactly for lower $d$, which was completed by several authors for $d=2$. In [5], Horozov & Iliev put any cubic Hamiltonian $H(x,y)$ having a critical point of a center type at the origin via affine change of variables into a normal form

$$H(x,y)=\frac{1}{2}x^2+\frac{1}{2}y^2-\frac{1}{3}x^3+axy^2+\frac{1}{3}by^3. \tag{2}$$

A series of results achieved $Z(2,2)=2$ by Horozov & Iliev [5], Gavrilov [3], Li & Zhang [9]. Petrov [10] found $Z(2,n)\leqslant n-1$ for $H(x,y)=y^2+x^3-x$. Gavrilov [2] gave $Z(2,n)\leqslant[(2/3)(n-1)]$ for $H(x,y)=(1/2)(x^2+y^2)-(1/3)x^3+xy^2$. Horozov & Iliev [6] concluded $Z(2,n)\leq 5n+15$ for system (2).

The main results about $Z(d,n)$ for $d=3$ is for elliptic Hamiltonian $H(x,y)=y^2/2+U(x)$ where $U(x)$ is a real polynomial in $x$ of degree at most four. For some cases with certain symmetric properties, such as $U(x)=x^4-x^2, U(x)=x^2-x^4$ and $U(x)=x^3-x$, $Z(d,n)$ has been estimated by Petrov [10–12]. For general case, Zhao & Zhang [14] divided $H(x,y)=y^2/2+U(x)$ $(\deg U(x)=4)$ into four normal forms and stated $Z(3,n)\leqslant 7n+5$.

There are few results about $Z(d,n)$ for $d\geqslant 3$ because of the complicated structure of Abelian integrals $I(h)$, see [4,15]. In this paper, we consider a kind of quartic Hamiltonian systems $H(x,y)=x^2+y^2-x^4+ax^2y^2+y^4$ where $a>-2$ is a constant, which can be rewritten as the planar form

$$\begin{cases}\dot{x}=2y(1+ax^2+2y^2),\\ \dot{y}=-2x(1-2x^2+ay^2).\end{cases} \tag{3}$$

The system (3) for $a>-2$ has three critical points: the saddles $S_1(-\sqrt{2}/2,0), S_2(\sqrt{2}/2,0)$ and the center $O(0,0)$. The periodic orbits surrounded the center and bounded by a heteroclinic loop, see Fig. 1. The critical level corresponding to the center is $h_c=0$, and the critical levels corresponding to the saddles is $h_s=1/4$.

We denote by $B(n)$ the upper bound of the numbers of zeros of Abelian integral $I(h)$ for $h\in(0,1/4)$ and $n=\max\{\deg f,\deg g\}$. Our main result is the following theorem.

**Theorem 1.** *For the Hamiltonian*

$$H(x,y)=x^2+y^2-x^4+ax^2y^2+y^4=h, \tag{4}$$

*with* $a>-2,\ B(n)\leq 3[\frac{n-1}{4}]+12[\frac{n-3}{4}]+23$. *Moreover,* $B(1)=B(2)=0,\ B(3)=B(4)\leq 15$.

## 2. The algebraic structure of Abelian integral

We shall express the Abelian integral as a linear combination of several basic integrals with polynomial coefficients in this section.

**Proposition 1.** *For the Hamiltonian (4), if* $a>-2$, *then the Abelian integral* $I(h)$ *can be expressed as follows*

$$I(h)=\alpha(h)I_{01}+\beta(h)I_{03}+\gamma(h)I_{21}+\delta(h)I_{23}, \tag{5}$$

*where* $I_{i,j}(h)=\oint_{\Gamma_h}x^iy^jdx, \alpha(h),\beta(h),\gamma(h)$ *and* $\delta(h)$ *are real polynomials in* $h$, *and* $\deg\alpha(h)\leqslant[\frac{n-1}{4}]=p$, $\deg\beta(h)=\deg\gamma(h)\leqslant[\frac{n-3}{4}]=q$, $\deg\delta(h)\leqslant[\frac{n-1}{4}]-1=p-1$.

**Proof.** We split the proof into several steps.

(1) Suppose that $H(x,y)=h$ contains a family of closed ovals $\Gamma_h$ surrounding the origin as $h\in\Sigma=(0,1/4)$, then $\Gamma_h$ are symmetric with respect to the coordinate axes. Obviously, $I_{i,2m}(h)=I_{2l+1,2m+1}(h)\equiv 0$. So we only need to consider $I_{2l,2m+1}(h)$. Therefore, by Green formula we obtain

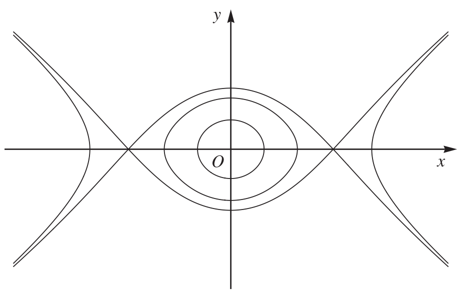


**Fig. 1.** Phase portrait of (3) for $a>-2$.