Applied Mathematics and Computation 228 (2014) 336–348

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Collocation methods for impulsive differential equations

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ARTICLE INFO

Keywords:
Impulsive differential equations
Collocation methods
Convergence
Superconvergence
Asymptotic stability

ABSTRACT

By choosing a suitable piecewise continuous collocation space, the convergence, global superconvergence and local superconvergence of the collocation solution for linear impulsive differential equations are derived. The conditions that the collocation solution is asymptotical stable are obtained and some numerical experiments are given.

1. Introduction

Impulsive differential equations appear to represent a natural framework for mathematical modelings of several real world phenomena. For instance, systems with impulse effects have applications in physics, biotechnology, industrial robotics, radiotechnology, pharmacokinetics, population dynamics, ecology, optimal control, the study of microorganism reproduction, economics, production theory, and so on. The investigation of impulsive differential equations first appears in 1960 (see [7]). As time goes on, the theories of the impulsive differential equations have been gradually improved. Many scientists have studied the impulsive differential equations from different angles (see [1,3,4]): global existence of solutions, the continuous dependence on parameter and the initial value, boundary value problems, periodic solution and so on. However, many impulsive differential equations cannot be solved analytically or their solving is more complication. At this time, taking numerical methods is a good choice. But to the best of our knowledge, there are few articles referred to this domain, see [8] (Runge–Kutta methods for linear impulsive differential equation), [6] (Runge–Kutta methods for linear impulsive differential systems), [5] (Lyapunov methods for nonlinear impulsive differential systems), [3] (Euler methods for impulsive delay differential equations), and so on.

In this paper, we use collocation methods to investigate the following impulsive differential equation:

$$\begin{cases} y'(t) = a(t)y(t), & t \neq \tau_k, \quad t \in I := [0, T], \\ \triangle y = B_k y, & t = \tau_k, \quad k = 0, 1, \ldots, \\ y(0^+) = y_0, \end{cases} \tag{1.1}$$

where $a : I \to \mathbb{R}$ is a given function and sufficient smooth, $\triangle y = y(t^+) - y(t), y(t^+)$ is the right limit of $y(t), 0 = \tau_{-1} < \tau_0 < \tau_1 < \cdots$. The paper is organized as follows. In Section 2, we give the collocation scheme and study the existence and uniqueness of collocation solutions. The global convergence of collocation solutions is investigated in Section 3, and in Section 4 we analyze their global and local superconvergence properties. Section 5 focuses on the asymptotic stability of the collocation solution. Section 6 contains the results of a sample of numerical experiments.

In the paper we always assume that there exist two constants θ_1 and θ_2 such that

$$0 < \theta_1 \leqslant \tau_k - \tau_{k-1} \leqslant \theta_2 < \infty \quad \text{for all } k \in \mathbb{Z}.$$

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http://dx.doi.org/10.1016/j.amc.2013.11.085

Definition 1.1. (Bainov [1]) The function $y : I \to \mathbb{R}$ is said to be a solution of the system (1.1) if the following conditions are satisfied:

1. $\lim_{t\to 0^+} y(t) = y_0 = y(0^+)$.
2. For $t \in I,\ t \neq \tau_k$, the function $y(t)$ is differentiable and $y'(t) = a(t)y(t)$.
3. The function $y(t)$ is left continuous in I, and if $t \in I$ and $t = \tau_k$, then $y(t^+) = (1 + B_k)y(t)$.

2. Collocation methods

Without loss of generality, we assume that $T = \tau_K$. In order to ensure the convergence of the method, we take τ_k as nodes. Let $p \geqslant 1$ be a given positive integer and $\tau_k = t_{(k+1)p} (-1 \leqslant k \leqslant K)$. On each intervals (τ_{k-1}, τ_k), we insert $p-1$ nodes and define as $t_{kp+1}, t_{kp+2}, \ldots, t_{kp+p-1}$. Let $h_n := t_{n+1} - t_n$ be the given stepsize on (t_n, t_{n+1}) and the mesh on I be defined by

$$I_h := \{t_n : 0 = t_0 < t_1 < \cdots < t_{(K+1)p} = T\}.$$

Accordingly, the collocation points are chosen as

$$X_h := \{t_{n,i} = t_n + c_i h_n : 0 < c_1 < \cdots < c_m \leqslant 1\},$$

where $\{c_i\}$ denotes a given set of collocation parameters.

Define $\sigma_n := (t_n, t_{n+1}]$. We approximate the solution by collocation in the piecewise polynomial space

$$\widetilde{S_m^{(0)}}(I_h) := \left\{ v : v|_{\bar{\sigma}_n} \in \pi_m, \begin{cases} \triangle v &= 0, \quad \text{if } t \neq \tau_k,\ t \in I \\ \triangle v &= B_k v,\ t = \tau_k \end{cases} \right\},$$

where π_m denotes the set of all real polynomials of degree not exceeding m, and $\triangle v := v(t^+) - v(t)$. Note that the usual piecewise polynomial space is $S_m^{(0)}(I_h) := \{v \in C(I) : v|_{\bar{\sigma}_n} \in \pi_m\}$. The collocation solution u_h is the element in this space that satisfies the collocation equation

$$\begin{cases} u_h'(t) &= a(t)u_h(t), \qquad t \neq \tau_k,\ t \in X_h, \\ \triangle u_h(t_{kp}) &= B_{k-1}u_h(t_{kp}), \quad k = 1, 2, \ldots, K+1, \\ u_h(0^+) &= y_0, \end{cases} \tag{2.1}$$

here, we provide $u_h(t)$ and $u_h'(t)$ is left continuous.

Now, let us consider the numerical solution of the initial value problem (1.1). By $u_h \in \widetilde{S_m^{(0)}}(I_h)$, we have

$$u_h'(t_n + v h_n) = \sum_{j=1}^{m} L_j(v) Y_{n,j}, \tag{2.2}$$

where $Y_{n,j} := u_h'(t_n + c_j h_n), L_j(v)$ is the Lagrange fundamental with respect to the distinct collocation parameters $\{c_i\}$ and the value is

$$L_j(v) := \prod_{i=1, i\neq j}^{m} \frac{v - c_i}{c_j - c_i}.$$

Integrating (2.2), we have

$$u_h(t_n + v h_n) = u_h(t_n^+) + h_n \sum_{j=1}^{m} \beta_j(v) Y_{n,j}, \quad v \in (0, 1], \tag{2.3}$$

where

$$\beta_j(v) := \int_0^v L_j(s)\, ds.$$

By the definition of $\widetilde{S_m^{(0)}}(I_h)$, we can obtain

$$u_h(t_n^+) = \begin{cases} u_h(t_n), & n \neq pk \\ (1 + B_{k-1})u_h(t_n), & n = pk, \end{cases} \quad k = 1, 2, \ldots K+1. \tag{2.4}$$

By (2.1) and (2.3), we have at $t_{n,i}$,

$$Y_{n,i} = a(t_{n,i})u_h(t_{n,i}) = a(t_{n,i}) \left[u_h(t_n^+) + h_n \sum_{j=1}^{m} a_{ij} Y_{n,j} \right],$$

where $a_{ij} := \beta_j(c_i)$.