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A path integration algorithm for stochastic structural dynamic systems

Yingguang Wang

School of Naval Architecture, Ocean & Civil Engineering, Shanghai Jiao Tong University, Shanghai 200240, China

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ABSTRACT

A numerical integration scheme based on closed Newton–Cotes formulas is first proposed for calculating the response statistics of a dynamic system under Gaussian white noise excitation. Accurate system response results down to very low probability levels have been obtained by utilizing the high-degree closed Newton–Cotes formulas for the numerical integration algorithm. The computational efficiency of the numerical integration scheme is found to be very high. The proposed Newton–Cotes scheme is also compared with the Gauss–Legendre scheme in the existing literature. It is found that the Newton–Cotes algorithm is easy to develop because Newton–Cotes quadrature uses values of the integrand at equally-spaced abscissas and the values of the quadrature coefficients are easily calculated.

1. Introduction

The response statistics of a linear structural system under additive Gaussian white noise excitation is known to be Gaussian. The response of a multidimensional nonlinear system subject to additive and/or multiplicative Gaussian white noise excitations is a Markov vector process. During the past several decades, various numerical methods have been developed for calculating the responses of structural systems under stochastic excitations. The numerical path integration solution is one such method that has been developed for predicting the responses of both linear and nonlinear systems. In the numerical path integral solution, efforts have been made by researchers to employ various kinds of interpolation procedures in order to increase the numerical efficiency. Wehner and Wolfer [1] used a piecewise constant interpolation scheme and their numerical method had predicted a too low peak and somewhat higher tail in the probability distribution. After a certain point this discrepancy could not be further reduced by choosing smaller time intervals. Wehner and Wolfer suggested that the numerical accuracy could be improved by employing a better interpolation procedure. Another kind of numerical path integral developed by Hsu and Chiu [2,3] is called a cell mapping method, and their basic idea is to consider the state space not as a continuum but rather as a collection of a large number of state cells with each cell being taken as a state entity. This method is similar to path integration with a piecewise constant interpolation scheme. The computation is quite intensive when using this method. Although short-time Gaussian approximation of the transition probability density has been used in the generalized cell mapping method [4] in order to improve the computing efficiency, a very large number of fine cells still need to be divided in a specific computation domain in order to improve the accuracy of the calculation results. In order to get system response results with high accuracy, Naess and Johnson [5,6] proposed another numerical scheme, and they employed a cubic B-spline interpolation procedure to increase the numerical efficiency. Marginal probability density values accurate to the order of 10^{-10} had been achieved [7], and this is very important for the prediction of the extreme responses. However, as we will see in the paper, there is still much room for the improvement of the accuracy of the response statistics by utilizing other numerical schemes. One such scheme is a Gauss–Legendre integration scheme [8,9] in

E-mail address: wyg110@sjtu.edu.cn

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which an implicit interpolation procedure has been employed. The essence of this integration scheme is that the values of the system response probability density are calculated at the Gauss quadrature points in sub-intervals, and the desired accuracy can be achieved with enough Gaussian points. However, Gauss–Legendre quadrature uses values of the integrand at oddly-spaced abscissas and the values of the quadrature coefficients are somewhat not easily calculated. Therefore, the Gauss–Legendre integration algorithm is somewhat complicated. Meanwhile, careful attention should be paid to the transformation of an arbitrary interval of integration to a standard one when using the Gauss–Legendre quadrature. The Newton–Cotes quadrature integration scheme proposed in this paper has the advantage of being able to obtain the system response results as highly accurate as the ones obtained by using the Gauss–Legendre scheme. Meanwhile, the Newton–Cotes algorithm is easy to develop because Newton–Cotes quadrature uses values of the function at equally-spaced points and the values of the quadrature coefficients are easily calculated.

2. Theoretical basis of the path integral solution

A Langevin equation of the type [10],

$$\dot{x}_i + \sum_{j=1}^{N} \gamma_{ij} x_j = F_i(t); \quad i = 1, 2, \ldots, N \tag{1}$$

with δ-correlated Gaussian distributed Langevin forces [10]

$$E[F_i(t)] = 0, \quad E[F_i(t)F_j(t')] = q_{ij}\delta(t - t'), \quad q_{ij} = q_{ji} \tag{2}$$

describes a process which is called an Ornstein–Uhlenbeck process. In Eq. (1) x_i is the system response state variable and γ_{ij} is a coefficient of constant. The essential feature is that the homogeneous equations (1) are linear and that the coefficients q_{ij} describing the strength of the noise do not depend on the variable x_k. It should be noted that with a vanishing matrix γ_{ij} ($\gamma_{ij} = 0$) the process described by (1), (2) is called a Wiener process. For one stochastic variable x, the general Langevin equation has the form [10]:

$$\dot{x} = h(x,t) + g(x,t)F(t) \tag{3}$$

where h and g are functions of variables x and t. The Langevin force $F(t)$ is again assumed to be a Gaussian random variable with zero mean and δ correlation function. For constant g, (3) is called a Langevin equation with an additive noise force. For g depending on x one speaks of a Langevin equation with a multiplicative noise term. The process described by (3) with δ correlated Langevin forces is a Markov process, i.e. its conditional probability at time t_n depends only on the value $x(t_{n-1}) = x_{n-1}$ at the next earlier time. Usually a formal general solution of the stochastic differential equation (3) cannot be given. However, we can set up a Fokker–Plank equation by which the probability density of the stochastic variable can be calculated. The Fokker–Plank equation for one variable x has the form [10]:

$$\frac{\partial p(x,t|x',t')}{\partial t} = \left[-\frac{\partial}{\partial x} D^{(1)}(x,t) + \frac{\partial^2}{\partial x^2} D^{(2)}(x,t) \right] p(x,t|x',t'). \tag{4}$$

In (4) $D^{(1)}(x)$ is called the drift coefficient and $D^{(2)}(x)$ the diffusion coefficient. The conditional probability $p(x,t|x',t')$ is called a transition probability density. Mathematically, Eq. (4) is a linear second-order partial differential equation of parabolic type. In [10] an expression for the transition probability density has been developed for a short time step τ up to the corrections of the order τ^2 in the following form:

$$p(x,t+\tau|x',t) = \frac{1}{2\sqrt{\pi D^{(2)}(x',t)\tau}} \exp\left[-\frac{[x - x' - D^{(1)}(x',t)\tau]^2}{4D^{(2)}(x',t)\tau} \right]. \tag{5}$$

The above transition probability density is needed for the path integral solutions. By repeatedly applying the Chapman–Kolmogorov equation

$$p(x_3,t_3|x_1,t_1) = \int p(x_3,t_3|x_2,t_2)p(x_2,t_2|x_1,t_1)dx_2 \tag{6}$$

we can express the evolution of $p(x, t)$ from the initial distribution $p(x_0, t_0)$ in terms of the transition probability density. Dividing the time difference $t - t_0$ into N small time intervals of length $\tau = (t - t_0)/N$, we have ($t_n = t_0 + n\tau$):

$$p(x,t) = \int dx_{N-1} \int dx_{N-2} \ldots \int dx_0 p(x,t|x_{N-1},t_{N-1})p(x_{N-1},t_{N-1}|x_{N-2},t_{N-2}) \ldots p(x_1,t_1|x_0,t_0)p(x_0,t_0). \tag{7}$$

For $N \to \infty$ we may use for the transition probability function the expression (5) for small τ, which then gives correct expectation values of $p(x, t)$ in the limit $N \to \infty$. Inserting (5) into (7) and taking the limit $N \to \infty$ we obtain with $x_N = x$, $[\tau = (t - t_0)/N]$