Applied Mathematics and Computation 228 (2014) 454–462

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

An iteratively approximated gradient projection algorithm for sparse signal reconstruction

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ARTICLE INFO

Keywords:
Sparse signal reconstruction
Quadratic programming
Gradient projection methods
Nonnegative constraints
Quasi-Lagrangian function
Partial duality

ABSTRACT

In this paper we developed an iteratively approximated gradient projection algorithm for ℓ_1-minimization problems arising from sparse signal reconstruction in compressive sensing. By introducing a relaxed variable, the noisy problem can be transformed into the problem with equality constraints. The nonsmooth ℓ_1 term was tackled by variable-splitting techniques. Thus the problem was transformed into a quadratic programming problem. All linear variables in the objective function were imposed on ℓ_2 regularization. Based on ideas of quasi-Lagrangian functions and partial duality, a reduced quadratic programming problem can be obtained iteratively. At each iteration, we applied gradient projection methods with approximated gradients to get the next iterates. The computational experiments show the proposed method is very effective.

1. Introduction

Finding sparse solutions to under-determined, or ill-conditioned linear systems of equations have arisen in digital signal processing [3,13,15,20,35], image deconvolution and denoising [10,18,33] and compressive sensing [6–8,24,32]. In order to having fewest nonzero elements, we can form the problem that finds sparse solutions into the following optimization problem

$$\min_x \|x\|_0, \quad s.t.\ Ax = b, \tag{1.1}$$

where A is a known $m \times n$ measurement matrix and b is the signal of observation, $\|x\|_0$ denotes the number of nonzero elements of vector x to be estimated.

Since this problem is NP-hard [14,27], (1.1) is often relaxed to the following optimization problem

$$\min_x \|x\|_1, \quad s.t.\ Ax = b, \tag{1.2}$$

where $\|x\|_1$ denotes the ℓ_1-norm of vector x. If the measurement matrix A and all the sparsest vectors satisfy some reasonable conditions, (1.2) shares common solutions with (1.1), see [6,9,13,15].

In practice, the observed signal b is usually contaminated by some unknown noises. In this case, an approximate reconstruction is preferable since the exact solution is impossible. Common relaxations to (1.2) include the constrained problem

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0096-3003/$ - see front matter
http://dx.doi.org/10.1016/j.amc.2013.10.063

$$\min_x \|x\|_1, \quad s.t.\ \|Ax-b\|_2 \leqslant \delta, \tag{1.3}$$

where $\|x\|_2$ denotes the ℓ_2-norm of $x \in \mathbb{R}^n$ and $\delta > 0$ is a tolerance measure of the noise. Closely related to the convex optimization problem (1.3) is the ℓ_1-regularization minimization problem

$$\min_x \|Ax-b\|_2^2 + \tau\|x\|_1, \tag{1.4}$$

where $\tau > 0$. The theory for penalty functions implies that the solution of the quadratic penalty function (1.4) goes to the solution of (1.3) as τ tends to zero.

In the last few years various methods have been proposed to efficiently solve problem (1.4). The most widely studied first-order method is the iterative shrinkage/thresholding (IST) method, which is first developed for wavelet-based image deconvolution [12,16,17,26,28,30,31]. The IST algorithm was also derived from an operator splitting framework and was combined with a continuation strategy, which leads to the fixed-point continuation algorithm (FPC) [21,22]. The TwIST is a two-step IST algorithm, in which the update equation depends on the two previous estimates [4]. FISTA is a fast IST algorithm which can attain the same optimal convergence in function values as Nesterov's multi-step gradient descent method [1]. The SpaRSA algorithm is another type of IST algorithms, which uses the Barzilai–Borwein(BB) steplength and an inner loop to guarantee the descent of the objective function sufficiently [34]. The GPSR algorithm is a gradient projection method which formulated the problem into a box-constrained quadratic programming and implemented a gradient projection with line search [19].

In the current paper, we transformed the noisy problem into an equality constrained problem by relaxed variables and impose the relaxed variables on ℓ_2-minimization and ℓ_1-minimization, respectively. Furthermore, the variable-splitting technique was used to make the objective function differentiable. Now the ℓ_2-minimization and ℓ_1-minimization problems are quadratic programming problems and linear programming problems, respectively. One quadratic regularization on all linear variables transformed both problems into two different quadratic programming problems. Based on ideas of quasi-Lagrangian function and gradient projection methods, we can solve the problem effectively.

2. Quadratic programming methods

Quadratic programming is the problem of minimizing a quadratic objective function subject to linear equality or inequality constraints. The general form is

$$\begin{aligned}
&\min_x \frac{1}{2}x^TQx + c^Tx,\\
&s.t.,\ Ax \leqslant b,\\
&\qquad Ex = d.
\end{aligned}$$

Here Q is a symmetric $n \times n$ matrix, x and c are column vectors with n elements. If there are only equality constraints, the quadratic programming problem may be approached relative easily. We can write the Lagrangian function as

$$L(x,\lambda) = \frac{1}{2}x^TQx + c^Tx + \lambda^T(Ex-d).$$

By seeking the extremum of the Lagrangian function with respective to Lagrange variable λ and the primal variable x, it may be shown that the solution to the equality constrained problem is given by the linear system

$$\begin{bmatrix} Q & E^T \\ E & 0 \end{bmatrix}\begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} -c \\ d \end{bmatrix}.$$

One special type of quadratic programming problems is with bound constraints, which reads as

$$\begin{aligned}
&\min_x \frac{1}{2}x^TQx + c^Tx,\\
&s.t.\ l \leqslant x \leqslant u.
\end{aligned}$$

This is also called box-constraint quadratic programming problem. The Karush–Kuhn–Tucker (KKT) condition of the above problem is

$$g_i(x) := (Qx+c)_i \begin{cases} < 0 & x_i = u_i, \\ = 0 & l_i < x_i < u_i, \\ > 0 & x_i = l_i. \end{cases} \tag{2.1}$$

One of the methods for solving box-constraint quadratic programming problems is gradient projection method, which has the following form