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Stability analysis of Runge–Kutta methods for systems $u'(t) = Lu(t) + Mu([t])$

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ABSTRACT

This paper deals with the stability of Runge–Kutta methods applied to the complex linear system $u'(t) = Lu(t) + Mu([t])$. The condition under which the numerical solution is asymptotically stable is presented, which is stronger than A-stability and weaker than A_f-stability. Furthermore, in the case of 2-norm and L being a real symmetric matrix, by using Padé approximation and order star theory, it is proved that for A-stable Runge–Kutta methods, suppose whose stability function is given by the (r,s)-Padé approximation to e^x, the numerical solution is asymptotically stable if and only if r is even.

1. Introduction

This paper deals with the stability of the numerical solution of the following differential equation with piecewise continuous argument(EPCA):

$$\begin{cases} u'(t) = Lu(t) + Mu([t]), & t \geqslant 0, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where $L, M \in \mathbb{C}^{d\times d}$, L is nonsingular, $u_0 \in \mathbb{C}^d$ is a given initial value, and $[\cdot]$ denotes the greatest integer function. The general form of EPCA is

$$\begin{cases} u'(t) = f(t, u(t), u(\alpha(t))), & t \geqslant 0, \\ u(0) = u_0, \end{cases} \tag{1.2}$$

where the argument $\alpha(t)$ has intervals of constancy. This kind of equations has been initiated by Wiener [26,28], Cooke and Wiener [4], and Shah and Wiener [21]. The general theory and basic results for EPCA have by now been thoroughly investigated in the book of Wiener [27]. The task of investigating EPCA is also of considerable applied interest since they include, as particular cases, impulsive and loaded equations of control theory and are similar to those found in some biomedical models.

Many real-life phenomena in physics, engineering, biology, medicine, economics, etc. can be modeled by a delay differential equation, and recently there are many literatures focused on this domain [1,2,5,9–11,14,16,30] etc. Particularly, in [30], the linear constant delay differential equation

$$u'(t) = Lu(t) + Mu(t - \tau) \tag{1.3}$$

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is studied, where $\|M\| < -\mu[L]$. It is said that the Runge–Kutta method with $h = \tau/m$ applied to (1.3) is stable if and only if it is A-stable. Moreover, in [1], the linear variable delay differential equation

$$u'(t) = au(t) + bu(t - \tau(t)) \tag{1.4}$$

is considered, where $|b| < -\Re(a)$. And it is said that the Runge–Kutta method applied to (1.4) is stable if and only if it is A_f-stable, but in fact, the set of A_f-stable Runge–Kutta methods is very meager.

In virtue of (1.1) being a special variable delay differential equation, the numerical stability condition between A-stability and A_f-stability is expected. In fact, the condition presented in Theorem 4.5 in the following shows it is stronger than A-stability and weaker than A_f-stability. Furthermore, it seems to us that the strong interest in differential equation with piecewise constant arguments is motivated by the fact that it describes hybrid dynamical system (a combination of continuous and discrete). These equations have the structure of continuous dynamical systems within intervals of unit length. Continuity of a solution at a point joining any two consecutive intervals implies recurrent relations for the values of the solution at such points. Therefore, they combine the properties of differential equations and difference equations.

There are also some authors who have considered the stability of numerical solutions for this kind of equations (see [7,15,17–20,22–24,29] etc.), but all of the above articles are based on real scaler problems.

Definition 1.1 (*Wiener [27]*). A solution of (1.1) on $[0,\infty)$ is a function $u(t)$ that satisfies the following conditions:

1. $u(t)$ is continuous on $[0,\infty)$.
2. The derivative $u'(t)$ exists at each point $t \in [0,\infty)$, with the possible exception of the point $[t] \in [0,\infty)$, where one-sided derivatives exist.
3. (1.1) is satisfied on each interval $[k, k+1) \subset [0,\infty)$ with integral endpoints.

Theorem 1.2 (*Wiener [27]*). *Problem (1.1) has on $[0,\infty)$ a unique solution $u(t) = M_0(\{t\})B_0^{[t]}u_0$, where $M_0(t) = e^{Lt} + (e^{Lt} - I)L^{-1}M, B_0 = e^L + (e^L - I)L^{-1}M$ and $\{t\}$ is the fractional part of t.*

2. The stability of the analytic solution

In this section we will give a sufficient condition under which the analytic solution of (1.1) is asymptotically stable.

Definition 2.1 (*asymptotic stability*). If any solution $u(t)$ of system (1.1) satisfies

$$\lim_{t\to\infty} u(t) = 0,$$

then the zero solution of system (1.1) is called asymptotically stable.

Lemma 2.2 (Wiener [27]). *The zero solution of system (1.1) is asymptotically stable if and only if $\rho(B_0) < 1$.*

In the paper, we always assume that $\|\cdot\|$ denotes the matrix norm induced by a vector norm on $\mathbb{C}^d$ and $\mu[\cdot]$ denotes the logarithmic norm of the matrix (see Dekker [6]), defined by

$$\mu[L] = \lim_{\triangle\to 0^+} \frac{\|I_d + \triangle L\| - 1}{\triangle},$$

where I_d is the $d \times d$ identity matrix.

Theorem 2.3. *The zero solution of system (1.1) is asymptotically stable if*

$$\begin{cases} (i)\ \mu[L] < 0, \\ (ii)\ \|M\| < -\mu[L]. \end{cases} \tag{1.1}$$

Proof. On one hand

$$\|B_0\| = \|e^L + (e^L - I)L^{-1}M\| \leqslant e^{\mu[L]} + \|(e^L - I)L^{-1}\|\|M\|.$$

On the other hand, by the condition (i), we know that $\mu[L] \neq 0$, so

$$\|(e^L - I)L^{-1}\| = \|\int_0^1 e^{Ls} ds\| \leqslant \int_0^1 \|e^{Ls}\| ds \leqslant \int_0^1 e^{\mu[L]s} ds = \frac{1}{\mu[L]}(e^{\mu[L]} - 1).$$

Noting that $e^{\mu[L]} - 1$ and $\mu[L]$ have the same sign, by the second condition $\|M\| < -\mu[L]$ in (1.1), we have