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On the implementation of the asymptotic Filon-type method for infinite integrals with oscillatory Bessel kernels ☆

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ABSTRACT

In this paper we consider the implementation of the asymptotic Filon-type method for the semi-infinite highly oscillatory Bessel integrals of the form $\int_1^\infty f(x)C_\nu(\omega x)\,dx$, where $C_\nu(\omega x)$ denotes Bessel function $J_\nu(\omega x)$ of the first kind, $Y_\nu(\omega x)$ of the second kind, $H_\nu^{(1)}(\omega x)$ and $H_\nu^{(2)}(\omega x)$ of the third kind, and modified Bessel function $K_\nu(\omega x)$ of the second kind, respectively, f is a smooth function on $[1,\infty)$, $\lim_{x\to\infty} f^{(k)}(x) = 0$ $(k = 0, 1, 2, \ldots)$ and ω is large. By approximating f by a linear combination of negative integer powers so that the moments can be expressed by some special functions, we complete the implementation of the method. Furthermore, we give the error analysis of the method for computing the integrals. The method is very efficient in obtaining very high precision approximations if ω is sufficiently large. Numerical examples are provided to confirm our analysis.

1. Introduction

In this paper we are concerned with the numerical evaluation of the integrals with a highly oscillatory Bessel kernel of the form

$$I[f] = \int_1^\infty f(x)C_\nu(\omega x)dx, \quad \omega \gg 1, \tag{1.1}$$

where f is a sufficiently smooth function on $[1,\infty)$, $\lim_{x\to\infty} f^{(k)}(x) = 0$ $(k = 0, 1, 2, \ldots)$, $C_\nu(\omega x)$ denotes Bessel function $J_\nu(\omega x)$ of the first kind, $Y_\nu(\omega x)$ of the second kind, $H_\nu^{(1)}(\omega x)$ and $H_\nu^{(2)}(\omega x)$ of the third kind, and modified Bessel function $K_\nu(\omega x)$ of the second kind, the order ν is arbitrary positive real number. For large ω, the integrand $f(x)C_\nu(\omega x)$ becomes highly oscillatory and presents serious difficulties in obtaining numerical convergence of the integrations (see [1–10]). This means that some general numerical methods may not be immediately applicable to the integrals. In the last decades, numerical approximations of $\int_a^b f(x)J_\nu(\omega x)dx$ $(0 \leqslant a < b < +\infty)$ have received a lot of attentions and have been the subject of a vast research by many authors (see, e.g., [5–21] and the references therein). The asymptotic method, Filon-type method and Levin-type method are among most important numerical methods. Nevertheless, for the case when the integration interval is unbounded, the literature is not so rich.

For $\int_a^b f(x)J_\nu(\omega x)dx$ and $0 \notin [a,b]$, the errors of the existing methods (the asymptotic method, Filon-type method and Levin-type method) are $O(\omega^{-n-\frac{3}{2}})$ [13]. It illustrates that these methods for computing this class of integrals have a common property: the errors decay as the parameter ω increases.

Recently, Shampine has taken up the basic algorithms on Filon method for approximating $\int_{-1}^1 f(x)e^{i\omega x}dx$ and discusses how they are used in an adaptive implementation [12]. Shampine has obtained the results by approximating $f(x)$ by using

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a truncated Chebyshev series expansion with the Chebyshev nodes and the Legendre polynomials with the Legendre nodes. The author has shown three approaches to error estimation.

In [17], a method for computing $\int_0^1 f(x)J_v(\omega x)dx$ is presented, which is based on the Filon-type method. By interpolating on $\left[\frac{j-1}{N},\frac{j}{N}\right]$ at distinct nodes $c_1=\frac{j-1}{N},\ c_2=\frac{2j-1}{2N},\ c_3=\frac{j}{N}$ by parabolic interpolation $P(x)$ and $c_1=\frac{j-1}{N},\ c_2=\frac{j}{N}$ by Hermite interpolation $S(x)$, respectively, where N is a positive integer and $j=1,2,\ldots,N$, and following [2], there exists

$$I[f]\approx\int_0^1 P(x)J_v(\omega x)dx,\quad I[f]\approx\int_0^1 S(x)J_v(\omega x)dx \tag{1.2}$$

with the errors

$$\left|I[f]-\int_0^1 P(x)J_v(\omega x)dx\right|\leqslant\frac{\sqrt{3}Ah^3}{36\sqrt[3]{\omega}}\max_{0\leqslant x\leqslant 1}|f^{(3)}(x)| \tag{1.3}$$

and

$$\left|I[f]-\int_0^1 S(x)J_v(\omega x)dx\right|\leqslant\frac{Ah^4}{384\sqrt[3]{\omega}}\max_{0\leqslant x\leqslant 1}|f^{(4)}(x)|, \tag{1.4}$$

where A is some constant and $h=\frac{1}{N}$.

In [15], a different method is explored for $\int_a^b f(x)J_v(\omega x)dx$ and $0\notin[a,b]$ which is based on a truncation of the asymptotic series. In the form

$$\int_a^b (0,f(x))^T\cdot(J_{v-1}(\omega x),J_v(\omega x))^T dx=\int_a^b F(x)\cdot W(\omega,x)dx, \tag{1.5}$$

where "·" denotes the inner product, since

$$W'(\omega,x)=\begin{pmatrix}\frac{v-1}{\omega} & -\omega\\ \omega & -\frac{v}{\omega}\end{pmatrix}W(\omega,x)=A(\omega,x)W(\omega,x), \tag{1.6}$$

then by letting $B(\omega,x)=\left(\frac{1}{\omega}A(\omega,x)\right)^T$ and

$$\begin{aligned}&F_1(x)=F(x)\\&F_{k+1}(x)=(B^T(\omega,x)F_k(x))',\quad k=1,2,\cdots,\end{aligned} \tag{1.7}$$

the asymptotic method is achieved by

$$\int_a^b f(x)J_v(\omega x)dx\approx\sum_{m=1}^{s}\frac{(-1)^{m+1}}{\omega^m}[F_m(x)\cdot B(\omega,x)W(\omega,x)]_a^b \tag{1.8}$$

with the error $O(\omega^{-s-3/2})$. In particular, for $\lim_{x\to\infty}f^{(k)}(x)=0\ (k=0,1,2,\ldots)$ and $b=\infty$, the asymptotic method can be defined simply as

$$\int_a^b f(x)J_v(\omega x)dx\approx\sum_{m=1}^{\infty}\frac{(-1)^{m}}{\omega^m}F_m(a)\cdot B(\omega,a)W(\omega,a). \tag{1.9}$$

In many physical applications, especially in the solution of certain mixed boundary value problems, we always encounter the integrals involving Bessel functions of the form

$$\int_0^\infty f(x)C_v(\omega x)dx. \tag{1.10}$$

Since $\int_0^\infty=\int_0^1+\int_1^\infty$ and $\int_0^1$ can be computed accurately by Xiang's ideas [17], then our key works are to compute the infinite integrals (1.1). In fact, for the case $\lim_{x\to\infty}f(x)=A$ (where A is a constant), the integrals (1.1) become

$$I[f]=\int_1^\infty[f(x)-A]C_v(\omega x)dx+A\int_1^\infty C_v(\omega x)dx. \tag{1.11}$$

Obviously, the function $f(x)-A$ satisfies the conditions of (1.1). From Eqs. 9.34–1(2/3) of [22], we have

$$x^kJ_v(\omega x)=x^kG_{0,2}^{1,0}\left(\frac{\omega^2x^2}{4}\middle|\begin{matrix} \\ \frac{v}{2},\quad -\frac{v}{2}\end{matrix}\right), \tag{1.12}$$

$$x^kY_v(\omega x)=x^kG_{1,3}^{2,0}\left(\frac{\omega^2x^2}{4}\middle|\begin{matrix}-\frac{v+1}{2}\\ -\frac{v}{2},\quad \frac{v}{2},\quad -\frac{v+1}{2}\end{matrix}\right), \tag{1.13}$$