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A remark on stochastic Logistic model with diffusion

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ABSTRACT

Stochastic Logistic system with diffusion is proposed and studied. Almost sufficient and necessary conditions for stability in time average and extinction are established. Some numerical examples are introduced to support the main results.

1. Introduction

In nature, the individuals of many species often move from high population densities to low to avoid congestion or crowding, for example arctic ground squirrels and vole species ([1]). Thus it is of great importance to investigate population models with diffusion. A classical logistic single-species system with diffusion can be denoted by

$$\frac{dx_i(t)}{dt} = x_i(t)[r_i - a_i x_i(t)] + \sum_{j=1,j\neq i}^{N} D_{ij}x_i(t)[x_j(t) - \alpha_{ij}x_i(t)], \quad i = 1,\ldots,N, \tag{1}$$

where x_i $(i = 1,2,\ldots,N)$ stands for the species x in patch $i, r_i > 0$ denotes the growth rate for species x in patch $i, a_i > 0$ is the self-inhibition coefficient, $D_{ij} \geqslant 0$ represents the dispersal coefficient of species x from patch j to patch i, $\alpha_{ij} \geqslant 0$ "corresponds to the boundary conditions of the continuous diffusion case, $\alpha_{ij} = 1$ for Neumann condition, $\alpha_{ij} \neq 1$ for Dirichlet or Robin conditions" ([2–6]), $i,j = 1,\ldots,N$. Allen [2] (Proposition 1) has proved that if $N = 2$ and $\Gamma := \bar{b}_1\bar{b}_2 - D_{12}D_{21} > 0$, then the unique positive equilibrium $x^* = (\Gamma_1/\Gamma, \Gamma_2/\Gamma)$ of system (1) is globally asymptotically stable (GAS), where $\bar{b}_1 = a_1 + D_{12}\alpha_{12}$, $\bar{b}_2 = a_2 + D_{21}\alpha_{21}$, $\Gamma_1 = r_1\bar{b}_2 + r_2D_{12}, \Gamma_2 = r_2\bar{b}_1 + r_1D_{21}$.

On the other hand, the growth of species in nature is inevitably affected by environmental noises. However, to the best of our knowledge, no results related to stochastic population system with diffusion have been reported.

In this report we study the parameter perturbation. May ([7]) has pointed out that due to environmental noises the growth rates in population systems should be stochastic (also see e.g., [8–23]). In practice, we usually estimate the growth rate r_i by an error term plus an average value. According to the famous central limit theorem the error term may be approximated by a normal distribution. At the same time, the growth rates may or may not correlate to each other. Taking these possible correlations into account, we hence use n independent Wiener processes $W_1(t), W_2(t), \ldots, W_n(t)$ to model the above normal distribution, e.g.,

$$r_i \to r_i + \sum_{j=1}^{n} \gamma_{ij}\dot{W}_j(t),$$

where γ_{ij}^2 is the intensity of the noise, $i = 1,\ldots,N, j = 1,\ldots,n$. Therefore Eq. (1) becomes the following Itô stochastic equation:

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$$dx_i(t) = x_i(t)[r_i - a_i x_i(t)]\, dt + \sum_{j=1, j\neq i}^{N} D_{ij} x_i(t)[x_j(t) - \alpha_{ij} x_i(t)]\, dt + \sum_{j=1}^{n} \gamma_{ij} x_i(t) dW_j(t), \tag{2}$$

Remark 1. Here, we use an n-dimensional Wiener process $(W_1(t), \ldots, W_n(t))$ to model the random noises, because the noise on r_i may or may not correlate to each other. If the noises are independent, we may choose $\gamma_{ii} \neq 0$ and $\gamma_{ij} = 0$ for $j \neq i$. If we choose $\gamma_{ij} \neq 0$, then the noises are correlate.

As said above, if $\Gamma > 0$, then the unique positive equilibrium x^* of system (1) is GAS. When system (1) is affected by noises, model (2) has no positive equilibrium. Consequently the solution of (2) never tend to a positive point. An interesting and important question arises naturally: whether (2) still keeps some stability around some positive point.In this report, we are going to show that (2) is stable in time average.

Theorem 1. *Suppose $N = 2$ and $\Gamma > 0$. Define*

$$\widetilde{\Gamma}_1 = 0.5\bar{b}_2 \sum_{j=1}^{n} \gamma_{1j}^2 + 0.5 D_{12} \sum_{j=1}^{n} \gamma_{2j}^2, \quad \widetilde{\Gamma}_2 = 0.5\bar{b}_1 \sum_{j=1}^{n} \gamma_{2j}^2 + 0.5 D_{21} \sum_{j=1}^{n} \gamma_{1j}^2.$$

$$\mu_1 = 2r_1 \Big/ \sum_{j=1}^{n} \gamma_{1j}^2, \quad \mu_2 = 2r_2 \Big/ \sum_{j=1}^{n} \gamma_{2j}^2, \quad v_1 = \Gamma_1 / \widetilde{\Gamma}_1, \quad v_2 = \Gamma_2 / \widetilde{\Gamma}_2.$$

(I) If $\mu_1 > \mu_2$ (which indicates $\mu_1 > v_2$), then we have
(i) If $v_2 > 1$, then both x_1 and x_2 are stable in time average almost surely (a.s.), i.e.,

$$\lim_{t\to+\infty} t^{-1} \int_0^t x_1(s) ds = \frac{\Gamma_1 - \widetilde{\Gamma}_1}{\Gamma}, \quad \lim_{t\to+\infty} t^{-1} \int_0^t x_2(s) ds = \frac{\Gamma_2 - \widetilde{\Gamma}_2}{\Gamma}, a.s. \tag{3}$$

(ii) If $v_2 < 1 < \mu_1$, then $\lim_{t\to+\infty} x_2(t) = 0$ a.s. and

$$\lim_{t\to+\infty} t^{-1} \int_0^t x_1(s) ds = \frac{r_1 - \sum_{j=1}^n 0.5\gamma_{1j}^2}{\bar{b}_1}, \ a.s.; \tag{4}$$

(iii) If $\mu_1 < 1$, then $\lim_{t\to+\infty} x_i(t) = 0$ a.s., $i = 1, 2$;
(II) If $\mu_1 < \mu_2$ (which indicates $\mu_2 > v_1$), then we have
(iv) If $v_1 > 1$, then

$$\lim_{t\to+\infty} t^{-1} \int_0^t x_1(s) ds = \frac{\Gamma_1 - \widetilde{\Gamma}_1}{\Gamma}, \quad \lim_{t\to+\infty} t^{-1} \int_0^t x_2(s) ds = \frac{\Gamma_2 - \widetilde{\Gamma}_2}{\Gamma}, a.s.$$

(v) If $v_1 < 1 < \mu_2$, then $\lim_{t\to+\infty} x_1(t) = 0$ a.s. and

$$\lim_{t\to+\infty} t^{-1} \int_0^t x_2(s) ds = \frac{r_2 - \sum_{j=1}^n 0.5\gamma_{2j}^2}{\bar{b}_2}, \ a.s.;$$

(vi) If $\mu_2 < 1$, then $\lim_{t\to+\infty} x_i(t) = 0$ a.s., $i = 1, 2$.

2. Proof

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space. Define

$$R_+^2 = \{a = (a_1, a_2) \in R^2 | a_i > 0, i = 1, 2\}, \quad \langle f(t) \rangle = t^{-1} \int_0^t f(s) ds, \quad f^* = \limsup_{t\to+\infty} f(t), \quad f_* = \liminf_{t\to+\infty} f(t).$$

Lemma 1 (Liu et al. [13]). *Suppose that* $z(t) \in C(\Omega \times [0, +\infty), R_+ = (0, +\infty))$.

(I) If there are two positive constants T and λ_0 such that $\ln z(t) \leqslant \lambda t - \lambda_0 \int_0^t z(s) ds + \sum_{i=1}^n \gamma_i W_i(t)$ for all $t \geqslant T$, where $\gamma_i, i = 1, \ldots, n$, are constants, then

$$\begin{cases} \limsup_{t\to+\infty} \langle z(t) \rangle \leqslant \lambda/\lambda_0 \ a.s., & \text{if } \lambda \geqslant 0; \\ \lim_{t\to+\infty} z(t) = 0 \ a.s., & \text{if } \lambda < 0, \end{cases}$$

(II) If there are three positive constants T, λ and λ_0 such that $\ln z(t) \geqslant \lambda t - \lambda_0 \int_0^t z(s) ds + \sum_{i=1}^n \gamma_i W_i(t)$ for all $t \geqslant T$, then $\liminf_{t\to+\infty} \langle z(t) \rangle \geqslant \lambda/\lambda_0$ a.s.