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On a new generalized inverse for matrices of an arbitrary index

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ABSTRACT

The purpose of this paper is to introduce a new generalized inverse, called DMP inverse, associated with a square complex matrix using its Drazin and Moore–Penrose inverses. DMP inverse extends the notion of core inverse, introduced by Baksalary and Trenkler for matrices of index at most 1 in (Baksalary and Trenkler (2010) [1]) to matrices of an arbitrary index. DMP inverses are analyzed from both algebraic as well as geometrical approaches establishing the equivalence between them.

1. Introduction and background

The symbol $\mathbb{C}^{m\times n}$ stands for the set of $m\times n$ complex matrices. The symbols $A^*, \mathcal{C}(A)$ and $\mathcal{N}(A)$ will denote the conjugate transpose, column space and null space of a matrix $A\in\mathbb{C}^{m\times n}$, respectively. Moreover, I_n will refer to the identity matrix of order n. If S and T are two complementary subspaces in $\mathbb{C}^{n\times 1}$ (that is, if $\mathbb{C}^{n\times 1}$ is direct sum of S and T) then the oblique projector onto S along T will be indicated by $P_{S,T}$. For a given matrix $A\in\mathbb{C}^{n\times n}$, this notation will be reduced to P_A when $S=\mathcal{C}(A)$ and T is the subspace orthogonal to S.

Let $A\in\mathbb{C}^{m\times n}$. The symbol $A^\dagger$ stands for the Moore–Penrose inverse of A, i.e., the unique matrix satisfying the following four Penrose conditions: $AA^\dagger A=A,\ A^\dagger AA^\dagger=A^\dagger,\ AA^\dagger=(AA^\dagger)^*,\ A^\dagger A=(A^\dagger A)^*$. A matrix X that satisfies the equality $AXA=A$ is called a g-inverse of A and if X satisfies $XAX=X$ it is called an outer inverse of A.

For a given matrix $A\in\mathbb{C}^{n\times n}$, recall that the smallest nonnegative integer m such that $rank(A^m)=rank(A^{m+1})$ is called the index of A and is denoted by $ind(A)$. The Drazin inverse of $A\in\mathbb{C}^{n\times n}$ is the unique matrix $A^d\in\mathbb{C}^{n\times n}$ such that $A^dAA^d=A^d,\ AA^d=A^dA,\ A^{m+1}A^d=A^m$, where $m=ind(A)$. If $A\in\mathbb{C}^{n\times n}$ satisfies $ind(A)\leqslant 1$, then the Drazin inverse of A is called the group inverse of A and is denoted by $A^\sharp$. When it exists, $A^\sharp$ is characterized as the unique matrix satisfying the conditions: $AA^\sharp A=A,\ A^\sharp AA^\sharp=A^\sharp,\ AA^\sharp=A^\sharp A$. For a given matrix $A\in\mathbb{C}^{n\times n}$, the symbol ${}^dC_A$ denotes its core-part, that is, ${}^dC_A=AA^dA$ [4]. The core-part of a matrix A does not reveal any interesting information when $ind(A)\leqslant 1$ because in this case it coincides with A.

We also recall that a square matrix A is called EP if $AA^\dagger=A^\dagger A$. Clearly, A is EP if and only if $A^\sharp=A^\dagger$, that is EP matrices have index at most 1 [5].

The core inverse of a matrix $A\in\mathbb{C}^{n\times n}$ is the unique matrix $A^{\circleddash}$ such that $AA^{\circleddash}=P_A$ and $\mathcal{C}(A^{\circleddash})\subseteq\mathcal{C}(A)$ [2,9].

While the Moore–Penrose and Drazin inverses of a matrix always exist, the group inverse as well as the core inverse of a square matrix A exist if and only if A and A^2 have the same rank, i.e., A is of index at most 1.

Some properties for all these generalized inverses can be found in [2–4,7,8]. All of these generalized inverses are known to be used in important applications. For example, the Moore–Penrose inverse is used to solve the least-squares problem, the group inverse has applications in Markov chain theory, the Drazin inverse gives the solution of a singular linear control

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system and the core inverse has applications in partial order theory (see for example [2–4,6]). As the core inverse is the unique least squares g-inverse, it may also be used in some least-squares problems and since the core inverse is also a ρ-inverse, it can be used in solving some difference equations.

The aim of this paper is to introduce and analyze a new generalized inverse, namely the DMP inverse, for a square matrix A of an arbitrary index m using the Drazin A^d and the Moore–Penrose $A^\dagger$ inverses of A. First, a canonical form for the DMP inverse is established and we conclude that our generalized inverse extends the core inverse introduced in 2010 by Baksalary and Trenkler in [2] in the sense that, while they necessarily require $m \leqslant 1$, in our case is not necessary. Furthermore, the equivalence of the algebraic definition (Definition 2.3) and the geometrical approach (Theorem 2.13) has been stated. We finally give some of the properties that this new generalized inverse possesses.

2. A new generalized inverse

We start this section by defining a new generalized inverse of a square matrix of an arbitrary index. In order to do this, we use the Drazin inverse (D) and the Moore–Penrose (MP) inverse and therefore we name this new generalized inverse as DMP inverse.

Let $A \in \mathbb{C}^{n\times n}$ have index m and consider the system of equations

$$XAX = X, \quad XA = A^dA, \quad A^mX = A^mA^\dagger. \tag{1}$$

Theorem 2.1. *If system (1) has a solution then it is unique.*

Proof. Assume that X_1 and X_2 satisfy (1), that is $X_1AX_1 = X_1$, $X_1A = A^dA$, $A^mX_1 = A^mA^\dagger$, $X_2AX_2 = X_2$, $X_2A = A^dA$ and $A^mX_2 = A^mA^\dagger$. Then, using that A^dA is a projector and $AA^d = A^dA$ we get

$$X_1 = X_1AX_1 = A^dAX_1 = (A^dA)^mX_1 = (A^d)^mA^mX_1 = (A^d)^mA^mA^\dagger = (A^d)^mA^mX_2 = (A^dA)^mX_2 = A^dAX_2 = X_2AX_2 = X_2. \qquad \square$$

Theorem 2.2. *The system of Eqs. (1) is consistent and has a unique solution: $X = A^dAA^\dagger$.*

Proof. It is easy to see that $A^dAA^\dagger$ satisfies the three equations in system (1). Now, Theorem 2.1 gives the uniqueness. $\square$

Thus, for a given square matrix A, the matrix $A^dAA^\dagger$ is the unique matrix satisfying system of Eqs. (1).

Definition 2.3. *Let $A \in \mathbb{C}^{n\times n}$ be a matrix of index m (not necessarily $\leqslant 1$). The DMP inverse of A, denoted by $A^{d,\dagger}$, is defined to be the matrix*

$$A^{d,\dagger} = A^dAA^\dagger.$$

Remark 2.4. Note that the new generalized inverse $A^{d,\dagger}$ can be firstly seen as an extension of that introduced in [9, p. 97] for matrices of index $m \leqslant 1$. In Remark 2.9 we will present $A^{d,\dagger}$ as an extension of the core inverse.

We now give the canonical form for the DMP inverse of a square matrix A using the Hartwig–Spindelböck decomposition [5,1]. For any matrix $A \in \mathbb{C}^{n\times n}$ of rank $r > 0$ the Hartwig–Spindelböck decomposition is given by

$$A = U\begin{pmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{pmatrix}U^*, \tag{2}$$

where $U \in \mathbb{C}^{n\times n}$ is unitary, $\Sigma = diag(\sigma_1 I_{r_1}, \sigma_2 I_{r_2}, \ldots, \sigma_t I_{r_t})$ is a diagonal matrix, the diagonal entries σ_i being singular values of A, $\sigma_1 > \sigma_2 > \cdots > \sigma_t > 0, r_1 + r_2 + \cdots + r_t = r$ and $K \in \mathbb{C}^{r\times r}, L \in \mathbb{C}^{r\times(n-r)}$ satisfy $KK^* + LL^* = I_r$. Using this fact we compute the Drazin inverse of A of index m as follows: Let

$$X = U\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}U^* \tag{3}$$

be the Drazin inverse of A partitioned conformable to A. Then X satisfies $XAX = X, AX = XA$ and $A^{m+1}X = A^m$. The equation $XAX = X$ implies

$$X_1\Sigma KX_1 + X_1\Sigma LX_3 = X_1, \tag{4}$$

$$X_3\Sigma KX_1 + X_3\Sigma LX_3 = X_3, \tag{5}$$