Applied Mathematics and Computation 226 (2014) 3–9

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Integrability conditions of derivational equations of a submanifold of a generalized Riemannian space

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ARTICLE INFO

Keywords:
Generalized Riemannian space
Submanifold
Derivational formulas
Integrability conditions

ABSTRACT

The basic information on generalized Riemannian space GR_N and its submanifold X_M and normal bundle X^N_{N-M} are given. As introduced connections are generally asymmetric, it is possible to define four kinds of covariant derivative.

In the present work one considers integrability conditions of derivational equations obtained by using the 1st and the 2nd kind of covariant derivative. We obtain also the corresponding Gauss–Codazzi equations. Similar results using simultaneously the both kinds of covariant derivative are obtained.

1. Introduction

1.1. A generalized Riemannian space GR_N [2,3,10] is a differentiable manifold equipped with an asymmetric basic tensor $G_{ij}(x^1,\dots,x^N)$ (the components) where x^i are the local coordinates. The symmetric, respectively antisymmetric part of G_{ij} are H_{ij} and K_{ij}.

For the lowering and rasing of indices in GR_N one uses H_{ij}, respectively H^{ij}, where

$$(H^{ij}) = (H_{ij})^{-1}, \quad (det(H_{ij}) \neq 0). \tag{1.1}$$

Cristoffel symbols at GR_N are

$$\Gamma_{i.jk} = \frac{1}{2}(G_{ji,k} - G_{jk,i} + G_{ik,j}), \quad \Gamma^i_{jk} = H^{ip}\Gamma_{p.jk}, \tag{1.2}$$

where, for example, $G_{ji,k} = \partial G_{ji}/\partial x^k$. Based on the asymmetry of G_{ij}, it follows that the Cristoffel symbols are also asymmetric with respect to j,k in (1.2).

By equations

$$x^i = x^i(u^1,\dots,u^M) \equiv x^i(u^\alpha), \quad i = 1,..,N, \tag{1.3}$$

a submanifold X_M is defined in local coordinates. If $rank(B^i_\alpha) = M$ $(B^i_\alpha = \partial x^i/\partial u^\alpha)$ and

$$g_{\alpha\beta} = B^i_\alpha B^j_\beta G_{ij}, \tag{1.4}$$

X_M becomes $GR_M \subset GR_N$, with induced basic tensor (1.4), which is generally also asymmetric. Note that in the present work Latin indices $i,j,\dots$ take values $1,\dots,N$ and refer to the GR_N, while the Greek ones take values $1,\dots,M$ and refer to the GR_M.

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[1] Authors were supported by Project 144032 MNTR Serbia.

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http://dx.doi.org/10.1016/j.amc.2013.10.016

In the GR_M are valid the relations similar to (1.1) and (1.2). The symmetric part of $g_{\alpha\beta}$ is denoted with $h_{\alpha\beta}$, and antisymmetric one with $k_{\alpha\beta}$, where e.g.

$$h_{\alpha\beta} = B^i_\alpha B^j_\beta H_{ij}, \quad (h^{\alpha\beta}) = (h_{\alpha\beta})^{-1}. \tag{1.5}$$

Cristoffel symbols $\widetilde{\Gamma}_{\alpha\cdot\beta\gamma}, \quad \widetilde{\Gamma}^\alpha_{\beta\gamma} = h^{\alpha\pi}\widetilde{\Gamma}_{\pi\cdot\beta\gamma}$ are expressed by $g_{\alpha\beta}$ analogously to (1.2).

For the unit, mutually orthogonal vectors N^i_A, which are orthogonal to the GR_M too, we have [4,5,8,9,11]

$$H_{ij}N^i_A N^j_B = e_A\delta^A_B = h_{AB}, \quad e_A \in \{-1, 1\}, \quad H_{ij}N^i_A B^j_\alpha = 0, \tag{1.6}$$

where $A, B, \ldots \in \{M+1, \ldots, N\}$.

As it is known, the following relations between Cristoffel symbols of a generalized Riemannian space and its subspace are valid:

$$\widetilde{\Gamma}_{\alpha\cdot\beta\gamma} = \Gamma_{i\cdot jk}B^i_\alpha B^j_\beta B^k_\gamma + H_{ij}B^i_\alpha B^j_{\beta,\gamma}, \tag{1.7}$$

$$\widetilde{\Gamma}^\alpha_{\beta\gamma} = h^{\pi\alpha}\widetilde{\Gamma}_{\pi\cdot\beta\gamma} = h^{\pi\alpha}(\Gamma_{i\cdot jk}B^i_\pi B^j_\beta B^k_\gamma + H_{ij}B^i_\pi B^j_{\beta,\gamma}), \tag{1.8}$$

i.e.

$$\widetilde{\Gamma}^\alpha_{\beta\gamma} = h^{\pi\alpha}H_{pi}B^p_\pi(\Gamma^i_{jk}B^j_\beta B^k_\gamma + B^i_{\beta,\gamma}). \tag{1.8$'$}$$

The set of normals of the submanifold $X_M \subset GR_N$ make a normal bundle for X_M, and we note it X^N_{N-M}. One can introduce a metric tensor on X^N_{N-M} [1,11,12]

$$g_{AB} = G_{ij}N^i_A N^j_B, \tag{1.9}$$

which is asymmetric in a general case.

The symmetric part is

$$h_{AB} = H_{ij}N^i_A N^j_B \underset{(1.5)}{=} e_A\delta^A_B = h_{BA} = \begin{cases} e_A, & \text{A = B}, \\ 0, & \text{otherwise}. \end{cases}, \; e_A \in \{-1, 1\}. \tag{1.10}$$

If

$$(h^{AB}) = (h_{AB})^{-1},$$

we have

$$h^{AB} = e_A\delta^A_B = h_{AB} = h^{BA}.$$

On X^N_{N-M} one can define in two manners connection coefficients [1,5,11,12]

$$\underset{\substack{1\\2}}{\overline{\Gamma}^A_{B\mu}} = H_{ij}h^{AQ} = N^j_Q(N^i_{B,\mu} + \underset{\substack{qp\\pq}}{\Gamma^i}N^p_B B^q_\mu). \tag{1.11}$$

Being the coefficients $\Gamma, \widetilde{\Gamma}, \overline{\Gamma}$ non-symmetric in general, for a tensor, defined at points of GR_M, is possible define four kinds of covariant derivative. For example

$$\underset{\substack{1\\2\\3\\4}}{\nabla_\mu} t^{i\alpha A}_{j\beta B} \equiv t^{i\alpha A}_{j\beta B\underset{\substack{1\\2\\3\\4}}{|}\mu} = t^{i\alpha A}_{j\beta B,\mu} + \underset{\substack{mp\\pm\\mp}}{\Gamma^i_{pm}} t^{p\alpha A}_{j\beta B}B^m_\mu - \underset{\substack{mj\\mj\\jm}}{\Gamma^p_{jm}} t^{i\alpha A}_{p\beta B}B^m_\mu + \underset{\substack{\mu\pi\\\pi\mu\\\mu\pi}}{\widetilde{\Gamma}^\alpha_{\pi\mu}} t^{i\pi A}_{j\beta B} - \underset{\substack{\mu\beta\\\mu\beta\\\beta\mu}}{\widetilde{\Gamma}^\pi_{\beta\mu}} t^{i\alpha A}_{j\pi B} + \underset{\substack{1\\2\\1\\2}}{\Gamma^A_{P\mu}} t^{i\alpha P}_{j\beta B} - \underset{\substack{1\\2\\2\\1}}{\Gamma^P_{B\mu}} t^{i\alpha A}_{j\beta P}. \tag{1.12}$$

In this way four connection $\underset{\theta}{\nabla}, \theta \in \{1, \ldots, 4\}$, on $X_M \subset GR_N$ are defined. We shall note the obtained structures $(X_M \subset GR_N, \underset{\theta}{\nabla}, \; \theta \in \{1, \ldots, 4\})$.

2. First and second kind integrability conditions of derivational equations

2.1. Based on the Th. 1.1 in [5], for tangent vectors of a submanifold $X_M \subset GR_N$ are in force derivational equations

$$B^i_{\alpha\underset{\theta}{|}\mu} = \sum_P \underset{\theta}{\Omega}_{P\alpha\mu}N^i_P, \quad \theta \in \{1, 2\} \tag{2.1}$$

and, based on the Th. 2.1., for unit normals