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On stable manifolds for planar fractional differential equations

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ABSTRACT

In this paper, we establish a local stable manifold theorem near a hyperbolic equilibrium point for planar fractional differential equations. The construction of this stable manifold is based on the associated Lyapunov–Perron operator. An example is provided to illustrate the result.

1. Introduction

In recent years, fractional differential equations have attracted increasing interest due to the fact that many mathematical problems in science and engineering can be modeled by fractional differential equations, see e.g. [4,15,18]. Although several results on asymptotic behavior of fractional differential equations are already published (e.g. on stability theory [3,8], linear theory [1,12], Lyapunov exponents [1,9], etc.), the development of a qualitative theory for fractional differential equations is still in its infancy. One of the reasons for this fact might be that general nonlinear fractional differential equations do not generate semigroups and the well-developed qualitative theory for dynamical systems cannot be applied directly.

In this paper we prove a stable manifold theorem for hyperbolic equilibria of fractional differential equations. We construct Lipschitz manifolds for two dimensional systems and omit all the technicalities which come into play when dealing with smoothness issues and exponential dichotomies – both well-know techniques for the construction of invariant manifolds for classical differential equations – and focus on those aspects related to Mittag–Leffler functions which are new in the construction of stable manifolds for fractional differential equations. We define a Lyapunov–Perron operator, using a solution representation formula [1] which provides the link between solutions of the nonlinear system and its linearization at the hyperbolic equilibrium. The unique parameter dependent fixed point of this operator describes the set of all solutions near the fixed point which tend to zero when time tends to infinity. This set is called the stable manifold of the hyperbolic fixed point of the fractional differential equation and our main result in this paper is to show that this stable manifold is the graph of a Lipschitz continuous function.

The paper is organized as follows: in Section 2, we recall some fundamental results on fractional calculus and fractional differential equations. Section 3 is devoted to the main result of this paper about stable manifolds for planar fractional differential equations. In Section 4 we conclude by summarizing the achievements in this paper and pointing out immediate extensions and related research questions.

To conclude this introductory section, we introduce notation which is used throughout this paper. Let $\mathbb{R}_{\geqslant 0}$ denote the set of all nonnegative real numbers. For a Banach space $(X, \|\cdot\|)$, let $\left(C_\infty(X), \|\cdot\|_\infty\right)$ denote the space of all continuous functions $\xi : \mathbb{R}_{\geqslant 0} \to X$ such that

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$$\|\xi\|_\infty := \sup_{t\in\mathbb{R}_{\geqslant 0}} \|\xi(t)\| < \infty.$$

It is well known that $(C_\infty(X), \|\cdot\|_\infty)$ is a Banach space. Let $\mathbb{R}^2$ be endowed with the max norm, i.e. $\|x\| = \max(|x_1|, |x_2|)$ for all $x = (x_1, x_2)^{\mathrm{T}} \in \mathbb{R}^2$. For $r > 0$, we define $B_r(0) := \{x \in \mathbb{R}^2 : \|x\| \leqslant r\}$. For a Lipschitz continuous function $f : \mathbb{R}^2 \to \mathbb{R}^2$, define

$$\ell_f(r) := \sup_{x,y\in B_r(0)} \frac{\|f(x) - f(y)\|}{\|x - y\|}.$$

2. Fractional differential equations

We start this section by briefly recalling an abstract framework of fractional calculus and the corresponding planar fractional differential equations. We refer the reader to the books [4,7] for more details about the theory of fractional differential equations.

Let $\alpha > 0$ and $[a, b] \subset \mathbb{R}$. Let $f : [a, b] \to \mathbb{R}$ be a measurable function such that $f \in L_1(a, b)$, i.e. $\int_a^b |f(s)|\, ds < \infty$. Then, the *Riemann–Liouville integral operator of order* α is defined by

$$(I^\alpha_{a+} f)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha-1} f(t)\, dt \quad \text{for } x > a,$$

where the *Euler Gamma function* $\Gamma : (0, \infty) \to \mathbb{R}$ is defined as

$$\Gamma(\alpha) := \int_0^\infty t^{\alpha-1} \exp(-t)\, dt,$$

see e.g. [4]. The corresponding *Riemann–Liouville fractional derivative* is given by

$$(D^\alpha_{a+}) f(x) := (D^m I^{m-\alpha}_{a+} f)(x),$$

where $D = \frac{d}{dx}$ is the usual derivative and $m := \lceil \alpha \rceil$ is the smallest integer bigger or equal α. On the other hand, the *Caputo fractional derivative* ${}^C D^\alpha_{a+} f$ of a function $f \in C^m([a, b])$, which was introduced by Caputo (see e.g. [4]), is defined by

$$({}^C D^\alpha_{a+} f)(x) := (I^{m-\alpha}_{a+} D^m f)(x), \quad \text{for } x > a.$$

We refer the reader to [4, Chapters 2 and 3], for a discussion on the relation and also some advantages of the Caputo derivative in comparison to the Riemann–Liouville derivative. In this paper, we consider planar fractional differential equations involving the Caputo fractional derivative

$${}^C D^\alpha_{0+} x(t) = Ax(t) + f(x(t)), \tag{1}$$

where $\alpha \in (0, 1),\ A \in \mathbb{R}^{2\times 2}$ and $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a Lipschitz function in a neighborhood of the origin satisfying that

$$f(0) = 0 \quad \text{and} \quad \lim_{r\to 0} \ell_f(r) = 0. \tag{2}$$

Note that f fulfills condition (2) provided that f is C^1 in a neighborhood of the origin with $f(0) = 0$ and $Df(0) = 0$. Assume that for any initial value $x \in \mathbb{R}^2$, the initial value problem (1), $x(0) = x$, has a unique solution denoted by $\varphi(\cdot, x)$ which is defined on the whole interval $\mathbb{R}_{\geqslant 0}$. This is for instance the case if f is globally Lipschitz, see e.g. [10, Theorem 3.1].

For $f = 0$, system (1) reduces to a linear time-invariant fractional differential equation

$${}^C D^\alpha_{0+} x(t) = Ax(t). \tag{3}$$

As shown in [1], $E_\alpha(t^\alpha A)x$ solves (3) with the initial condition $x(0) = x$, where the *Mittag–Leffler matrix function* $E_\alpha(A)$ for a matrix $A \in \mathbb{R}^{2\times 2}$ is defined as

$$E_{\alpha,\beta}(A) := \sum_{k=0}^\infty \frac{A^k}{\Gamma(\alpha k + \beta)}, \quad E_\alpha(A) := E_{\alpha,1}(A).$$

If the nonlinear term f does not vanish, it is in general impossible to provide an explicit form of the solution of (1), $x(0) = x$. However, we get a presentation of solutions for (1) by using the Mittag–Leffler matrix function. We refer the reader to [1,7] for a proof of this result. See also [11] for solution representation.

Theorem 1 (Solution representation formula for fractional differential equations). *The solution* $\varphi(\cdot, x)$ *of* (1), $x(0) = x$, *satisfies for* $t \in \mathbb{R}_{\geqslant 0}$

$$\varphi(t, x) = E_\alpha(t^\alpha A)x + \int_0^t (t - s)^{\alpha-1} E_{\alpha,\alpha}((t - s)^\alpha A) f(\varphi(s, x))\, ds. \tag{4}$$

By virtue of the above formula, in order to investigate asymptotic behavior of solutions of (1), it is helpful to understand the asymptotic behavior of the *Mittag–Leffler function* $E_{\alpha,\beta} : \mathbb{R} \to \mathbb{R}$ defined as