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On the second order implicit difference schemes for a right hand side identification problem

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ABSTRACT

In many physical phenomena, especially in temperature over-specification partial differential equation with an unknown source function appears. The present paper is devoted to the study of the well-posedness of the approximate solution of a right-hand side identification problem for a parabolic equation. The second order of accuracy implicit difference scheme is presented. The coercive stability estimates for the solution of this difference scheme are obtained. The theoretical statements for the solution of this difference schemes are supported by results of numerical experiments.

1. Introduction

Inverse parabolic problems are of significant importance in mathematical sciences, applied sciences and engineering. In many physical phenomena, for instance in the process of transportation, diffusion and conduction of natural materials, the parabolic partial differential equation is induced (see [39]). In inverse problems, the optimal overdetermination conditions are analyzed in some classical boundary conditions or/and similar conditions given at a point. The problem of determining the temperature at one end of a rod from temperature measurements at an interior point is an example of an inverse heat conduction problem (IHCP) which has been extensively studied [1].

Considerable efforts have been expanded in formulating numerical solution methods that are both accurate and efficient. Methods of numerical solutions of parabolic problems with parameters have been studied by many researchers (see, [5,6,10,13,15–19,27,32,34,35,40–43,45]).

One usually focuses himself on the uniqueness and the stability of the inverse problem. For the uniqueness, we refer [21]. The discussion of the stability is preparatory to the numerical implementation for the inverse problem in the theoretical respect. During the last decades, some numerical techniques have been proposed to solve inverse problems (for instance, see [7,22,29,31,33]. Determination of a control function in three-dimensional parabolic equation [14,38] and in polar coordinate system (for example, see [44]) are also investigated. Among them the finite difference method and the finite element method are so far the principal numerical tool of choice for the modeling and simulation of the IHCP. Also, authors previously in [4] constructed first order of accuracy (Rothe) and second order of accuracy (Crank–Nicholson) difference schemes for numerically solving a parabolic inverse problem with the Dirichlet condition.

On the other hand, the well-posedness of the right-hand side identification problem for a parabolic equation where the unknown function p is in space variable is well-investigated in [2,11,12,20]. To author's knowledge, when the unknown function p is in time variable, there are also a few works (for instance, see [3,9,36,37]).

One application of inverse heat conduction problem in engineering and science is to predict the thermal conductivity from the measured temperature profiles. The inverse estimation of thermal conductivity by the measured temperature profiles has been studied by many researchers [9,23–25,28,30,46]. In the article [9], thermal conductivity is taken as a function

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of the spatial coordinates. This work is devoted to the study of the well-posedness of the approximate solution of the right-hand side identification problem

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = a(x)\frac{\partial^2 u(t,x)}{\partial x^2} - \sigma u(t,x) + p(t)q(x) + f(t,x), \\ 0 < x < l,\ \ 0 < t < T, \\ u(t,0) = u(t,l) = 0,\ \ 0 \leqslant t \leqslant T, \\ u(0,x) = \varphi(x),\ \ 0 \leqslant x \leqslant l, \\ u(t,x^*) = \rho(t),\ \ 0 < x^* < l,\ \ 0 \leqslant t \leqslant T \end{cases} \tag{1}$$

under the assumptions $q(0) = q(l) = 0$ and $q(x^*) \neq 0$. Here $u(t,x)$ and $p(t)$ are unknown functions, $f(t,x), q(x), \varphi(x), \rho(t)$ and $a(x)$ are given sufficiently smooth functions, $a(x) \geqslant \delta > 0$ and $\sigma > 0$ is a sufficiently large number.

For approximately solving the parabolic inverse problem (1), the second order implicit difference scheme is constructed. The well-posedness theorem for the solution of this difference scheme is established. The theoretical statements are supported by numerical experiments. A comparison of error analysis is given.

2. Second order of accuracy implicit difference scheme

For the approximate solution of the problem (1), the second order of accuracy implicit difference scheme

$$\begin{cases} \frac{u_n^k - u_n^{k-1}}{\tau} - a(x_n)\frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} + \sigma u_n^k \\ \qquad + \frac{\tau}{2}\Big(-a(x_n)\frac{1}{h^2}\Big(-a(x_{n+1})\frac{u_{n+2}^k - 2u_{n+1}^k + u_n^k}{h^2} + \sigma u_{n+1}^k \\ \qquad -2\Big(-a(x_n)\frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} + \sigma u_n^k\Big) - a(x_{n-1})\frac{u_n^k - 2u_{n-1}^k + u_{n-2}^k}{h^2} + \sigma u_{n-1}^k\Big) \\ \qquad +\sigma\Big(-a(x_n)\frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} + \sigma u_n^k\Big)\Big) \\ \quad = \frac{p^k + p^{k-1}}{2}\Big(q_n + \frac{\tau}{2}\Big(-a(x_n)\frac{q_{n+1} - 2q_n + q_{n-1}}{h^2} + \sigma q_n\Big)\Big) + f\left(t_k - \frac{\tau}{2}, x_n\right) \\ \qquad + \frac{\tau}{2}\left(-a(x_n)\frac{f\left(t_k - \frac{\tau}{2}, x_{n+1}\right) - 2f\left(t_k - \frac{\tau}{2}, x_n\right) + f\left(t_k - \frac{\tau}{2}, x_{n-1}\right)}{h^2} + \sigma f\left(t_k - \frac{\tau}{2}, x_n\right)\right), \\ p^k = p(t_k),\ \ q_n = q(x_n),\ \ x_n = nh,\ \ t_k = k\tau,\ \ 1 \leqslant k \leqslant N,\ \ 2 \leqslant n \leqslant M-2, \\ Mh = l,\ \ N\tau = T;\ \ u_0^k = u_M^k = 0,\ \ 0 \leqslant k \leqslant N, \\ u_n^0 = \varphi(x_n),\ \ 0 \leqslant n \leqslant M, \\ u_s^k + \frac{u_{s+1}^k - u_s^k}{h}(x^* - sh) = \rho(t_k),\ \ 0 \leqslant k \leqslant N, 0 < s = \left[\frac{x^*}{h}\right] < M \end{cases} \tag{2}$$

is presented. Here, $q_0 = q_M = 0, q_s, q_{s+1} \neq 0$ are assumed.

In the present paper, positive constants, which can be differ in time will be indicated with an M. On the other hand $M(\alpha, \beta, \ldots)$ is used to focus on the fact that the constant depends only on $\alpha, \beta, \ldots$.

To formulate our results, first we introduce some spaces. With the help of a positive operator A, we introduce the fractional spaces $E'_\alpha, 0 < \alpha < 1$, consisting of all $v \in E$ with the finite norm

$$\|v\|_{E'_\alpha} = \|v\|_E + \sup_{\lambda > 0} \lambda^\alpha \|A(\lambda + A)^{-1} v\|_E.$$

The Banach space $\overset{\circ}{C}{}_h^\alpha = \overset{\circ}{C}{}^\alpha[0,l]_h, \alpha \in (0,1)$, of all grid functions $\phi^h = \{\phi_n\}_{n=1}^{M-1}$ is defined on

$$[0,l]_h = \{x_n = nh, 0 \leqslant n \leqslant M, Mh = l\},$$

with $\phi_0 = \phi_M = 0$ equipped with the norm

$$\left\|\phi^h\right\|_{\overset{\circ}{C}{}_h^\alpha} = \left\|\phi^h\right\|_{C_h} + \sup_{1 \leqslant n < n+r \leqslant M-1} |\phi_{n+r} - \phi_n| (rh)^{-\alpha},$$

$$\left\|\phi^h\right\|_{C_h} = \max_{1 \leqslant n \leqslant M-1} |\phi_n|.$$

Moreover, $C_\tau(E) = C([0,T]_\tau, E)$ is the Banach space of all grid functions $\varphi^\tau = \{\varphi(t_k)\}_{k=1}^N$ defined on $[0,T]_\tau = \{t_k = k\tau, 0 \leqslant k \leqslant N, Nh = T\}$ with values in E equipped with the norm

$$\|\varphi^\tau\|_{C_\tau(E)} = \max_{1 \leqslant k \leqslant N} \|\varphi(t_k)\|_E.$$

Then, the following theorem on well-posedness of problem (2) is established.