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Geometry of the *Poincaré compactification* of a four-dimensional food-web system

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ABSTRACT

In this paper, the behavior of dynamics 'at infinity' of a four-dimensional autonomous food web system has been investigated. For this, a topological method has been developed to understand the geometry of the Poincaré compactification which investigate the behavior of the vector field at infinity. The global phase portrait has been shown on the *Poincaré disc*.

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1. Introduction

The projection of the real line on a circle is a type of compactification that was known to Greek mathematicians before the development of the theory of dynamical systems. The French mathematician and theoretical physicist Jules Henri Poincaré (1854 – 1912) began to study the qualitative aspects of systems of differential equations in the late nineteenth century. This analysis was a breakthrough in the field of dynamical systems because one no longer had to obtain a specific solution of the equation in question to understand its general behavior. This manner of analysis was introduced by Poincaré in his paper *Mémoire sur les courbes définies par une équation différentielle"* [1]. Poincaré worked primarily with systems of two variables and played a key role in identifying the existence of limit cycles with the Poincaré-Bendixson theorem. Poincaré also searched for a complete global analysis of a system of two variables; to do so he introduced analysis at infinity by means of the Poincaré Sphere. These two aspects of his analysis join in a behavior known as a *limit cycle at infinity* [2,3].

In 1881 Poincaré studied limit cycles at infinity of two dimensional polynomial differential equations via compactification. The main idea of this method is to identify $\mathfrak{R}^n$ with northern and southern hemispheres through simple projections, then the vector field X on $\mathfrak{R}^n$ can be extended to a vector field $\tilde{X}$ on S^n. This method is called the *Poincaré compactification*.

The study of solutions escaping to infinity has been an important tool in order to understand the global picture of a dynamical system in $\mathfrak{R}^n$. The compactification technique consists in writing the equations of motion as a vector field and then applying the *Poincaré compactification,* which is a method to extend analytically the vector field to a compact manifold, in fact to a sphere. This tool or method is very important to study the qualitative dynamic of the flow at infinity or in the unbounded part. From the behavior of the Poincaré sphere, one can construct the "global phase portrait" without knowing the exact analytic solutions [3,4].

Through the use of projective geometry, the complete behavior of a two dimensional system of differential equations can be seen in its behavior on a sphere of finite radius known as the Poincaré sphere. To do this, one places the phase plane tangent to the sphere and makes correspond points on the plane with the points on the sphere by central projection (a point on the plane corresponds with an antipodal pair on the sphere). One must note that the line intersects the sphere at two points;

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to remove this non-uniqueness, antipodal points are identified on the Poincaré sphere. The points that were at infinity on the original plane become points on the equator of the sphere [2,5].

From the behavior on the Poincaré sphere, one can construct the global phase portrait [6,7]. To consider the global phase portrait of a system, one projects the trajectories from the upper hemisphere orthogonally down onto the plane that goes through the center of the sphere and is parallel to the original plane. In this way the complete behavior on the original plane becomes the behavior on the finite disk of this new plane. Points at infinity become the points at which the sphere intersects the plane the boundary of the disk. The first analysis that one would do for a system in the finite plane is to find the location of the fixed points. With this as motivation, the primary problem of analyzing the behavior of a system at infinity is to determine the location of fixed points, if there are any.

Then, in each case they give a global expressions for the *Poincaré compactification.* As an application, and using the fact that the vector field of the n-body problem can always be written in the form of a polynomial vector field (see [3] for example), the *Poincaré compactifications* for the Kepler problem on the line and on the plane and for the collinear 3-body problem are computed. The main disadvantage here for obtaining this polynomial vector field is the use (in general) of redundant variables. Our purpose in this paper is to understand the geometry of the *Poincaré compactification* [see the Sections 1.1 and 1.2] and apply this technique in vector fields defined by rational functions. We will also give a global expressions for the Poincaré vector field associated to nonlinear differential equations.

The dynamics 'at infinity' has not been received much attention, even though it was central to Poincaré's analysis of qualitative dynamics. Before the development of the theory of dynamical systems, the qualitative approach involved defining dynamics on a compact state space that is in fact the projective plane [1]. The recent development of dynamical systems have less attention to the question of the pathological dynamics '*far away*'.

The aim of this paper is to understand the geometry of the *Poincaré compactification* of a four-dimensional continuous dynamical system originally developed by Gakkhar et. al., [8]. This is an atttempt to establish *Poincaré compactification* in four-dimensional food web systems.

1.1. Poincaré compactification

In [1], Poincaré studied the two dimensional polynomial vector fields on the plane $\mathfrak{R}^2$, by means of central projection of the paths on a sphere S^2, tangent to the plane at the origin [1]. Thus he provided the means for studying the behavior of the field on a neighborhood of infinity, which is represented by the equator S^1.

Let P_1 and P_2 be the polynomials of arbitrary degrees d_1 and d_2 in the variables x_1 and x_2. Consider the following polynomial vector field of degree $d = \max\{d_1, d_2\}$ in $\mathfrak{R}^2$:

$$X = P_1(x_1, x_2)\frac{\partial}{\partial x_1} + P_2(x_1, x_2)\frac{\partial}{\partial x_2}$$

In order to effect the Poincare construction we identify $\mathfrak{R}^2$ with the hyperplane $\{x \in \mathfrak{R}^3 : x_3 = 1\}$, tangent to the sphere

$$S^2 = \{y \in \mathfrak{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\} \tag{1}$$

at the north pole. The sphere S^2 defined in Eq. (1) is said to be *Poincaré sphere* and is tangent to $\mathfrak{R}^2$ at the point $(0, 0, 1)$.

The sets

$$\begin{aligned} S_+^2 &= \{y \in S^2 : y_3 > 0\} \\ S_-^2 &= \{y \in S^2 : y_3 < 0\} \\ \text{and} \quad S_1 &= \{y \in S^2 : y_3 = 0\} \end{aligned} \tag{2}$$

are called the northern hemisphere, the southern hemisphere and the equator respectively. Now, consider the projection of the vector field X from $\mathfrak{R}^2$ to S^2 is given by the central projections with the following two diffeomorphisms

$$\begin{aligned} f^+(x) &: \mathfrak{R}^2 \to S_+^2 \\ f^-(x) &: \mathfrak{R}^2 \to S_-^2 \end{aligned} \tag{3}$$

such that

$$f^{\pm}(x) = \pm\left(\frac{x_1}{\Delta(x)}, \frac{x_2}{\Delta(x)}, \frac{1}{\Delta(x)}\right), \quad \text{where} \quad \Delta(x) = \sqrt{x_1^2 + x_2^2 + 1}$$

These maps send (x_1, x_2) to one of the intersections of the line that join the origin with $(x_1, x_2, 1)$. The first map sends $\mathfrak{R}^2$ diffeomorphically onto the northern hemisphere S_+^2, and the second map sends it onto the southern hemisphere S_-^2. The equator S^1 corresponds to the infinity of $\mathfrak{R}^2$.

This induced an analytically conjugate vector field $\bar{X}(y)$ to vector field X in each hemisphere everywhere tangent to S^2 such as $\bar{X}(y) = Df^{\pm}(x)X(x)$. Notice that the points at infinity of $\mathfrak{R}^2$ are in bijective correspondence with the points of the equator of S^2. If the vector field is multiplied by a factor $\rho(x) = x_3^{d-1}$, then it is possible to extend the vector field from $S^2 \setminus S^1$ to S^2. The extended vector field on S^2 is called the *Poincaré compactification* of the vector field X.

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