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Error estimates for a class of partial functional differential equation with small dissipation ☆

Aditya Kaushik

Department of Mathematics, University Institute of Engineering and Technology, Panjab University, Chandigarh 160014, UT, India

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ABSTRACT

This article deals with a class of partial functional differential equation with a small dissipating parameter on a rectangular domain. Classical numerical methods for solving this type of problems reveal disappointing behavior or are tremendously expensive in computer memory and processor time. This arises because the precision of an approximate solution depends inversely on perturbation parameter values and, thus, it deteriorates as a parameter decreases. Therefore, it is of particular interest to develop numerical methods whose error estimates would be independent of the perturbation parameter contaminating the solution. In order to overcome the said difficulty we derive robust parameter uniform error estimates for a class of partial functional differential equations. The analysis presented in this paper uses a suitable decomposition of the error into smooth and a singular component combined with the appropriate barrier functions and comparison principle.

1. Introduction

In this paper we study the parameter uniform method for time delayed parabolic singular perturbation problem. This type of problems arise in a diverse area of science and engineering that take into account not just the present state of physical systems but also its past history; for instance, in studying heat or mass transfer [1–3], drift diffusion model of semiconductor devices [4], fluid flow [5], and biosciences [6–9]. The uppermost derivative in the equation is multiplied by a minuscule parameter. If the parameter vanishes, solution of these problems exhibits multiscale character and exhibits boundary layer character [5]. Boundary layers are rapidly varying solution components that have support in narrow regions close to the boundary of the domain [5,10]. This complexity of the system renders it unlikely to obtain an analytical solution and numerical solution of the problem would seem more practical. However, for the numerical solution of such problems, these layers are connected with additional difficulties; besides instabilities of certain discretization methods, high computational costs and insufficient resolution are essentially due to the existence of layers [10,11].

There is a wide variety of asymptotic and numerical methods available for solving singular perturbation problems; for example, method of matched asymptotic expansion [12,13], multiple scales [14,15], streamline-diffusion method [16], domain decomposition [17], collocation method [18], finite elements [19–21], and fitted mesh method [22,23]. For certain problems parameter uniform methods which converge irrespective of perturbation parameter have been developed and thoroughly studied [24–26] with the convergence analysis given in the maximum norm.

Solutions of delay differential equations are of immense interest, equally in applications and theory. An important example arises in the study of slowly oscillating solutions of scalar delay differential equation

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E-mail address: akaushik@kuk.ac.in

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$$\dot{\eta}(t) = \lambda\eta(t) + \lambda f(\eta(t-1)),$$

which was studied in [27–29]. Much attention has been paid to delay differential equations and their numerical approximations (see, e.g., [30] and the references therein) but not to the singularly perturbed delay differential equations. Among the first rigorous numerical treatments of singularly perturbed equations with delay is the pioneering work of Kadalbajoo et al. [31–34] in which second order singularly perturbed delay differential equations are approximated by finite differences on piecewise equidistant meshes. The fact that singular perturbation problems involve partial functional differential equation is of crucial importance as many additional technical issues arise with partial differential equations; such as, smoothness of the solution, compatibility of data and geometry of the domain. In fact, given the rich variety of physical, geometric and probabilistic phenomena which can be modeled by partial differential equation, the study of partial functional differential equations is of great practical value. But there is no general theory known concerning the solvability of partial functional differential equations with delay. Instead the research focuses on various particular partial differential equations that are important for applications within and outside of mathematics. Our aim in this paper is to carry out such a study and derive parameter uniform error estimates for a singular perturbed time delayed parabolic partial differential equation.

2. Continuous problem and auxiliary results

We consider the time delayed parabolic initial boundary value problem

$$\left.\begin{array}{l} \mathcal{L}_\epsilon u(x,t) := u_t(x,t) - \epsilon u_{xx}(x,t) + a u_x(x,t-\tau) + b u(x,t) = f(x,t) \\ u(x,s) = \varphi(x,s); \quad x \in \Omega \equiv [0,1],\ s \in [-\tau, 0), \\ u(x,0) = \varphi(x,0) = \varphi(x); \quad x \in \Omega \equiv [0,1], \\ u(0,t) = 0, \quad u(1,t) = 0;\ t \in (0,T], \end{array}\right\} \tag{2.1}$$

where $0 < \epsilon \ll 1$ is a small parameter and $0 \leqslant \tau = o(\epsilon^2)$ is the time delay ($\tau = T/k$ for some integer $k > 1$). Furthermore, we assume that $f(x,t), a(x,t)$ and $b(x,t)$ are sufficiently smooth functions which satisfies

$$0 < \beta \leqslant a(x,t) \text{ and } 0 < \eta \leqslant b(x,t),$$

where η and β are positive constant independent of ϵ.

When the delay values $t-\tau$ are bounded away from t by a positive constant the existence of the solution can be verified using the method of steps. For a general discussion of the properties enjoyed by the solution of parabolic differential equations we refer to [35]. The following two lemmas are immediate consequences of Theorem 3.2 of [36].

Lemma 2.1. *Consider the initial boundary value problem*

$$\left.\begin{array}{l} u_t(x,t) - \epsilon u_{xx}(x,t) + a u_x(x,t-\tau) + b u(x,t) = f(x,t) \quad \text{on } D = \Omega \times (0,T] \\ u = 0 \quad \text{on } \partial D, \end{array}\right\} \tag{2.2}$$

where a, b, and c are smooth functions on $\overline{D}$. Let $\alpha \in (0,1)$.

1. Suppose that $f \in C^{0,\alpha}(\overline{D})$. Then (2.2) has a solution $u \in C^{1,\alpha}(\overline{D}) \bigcap C^{2,\alpha}(D)$.
2. Suppose that $f \in C^{0,\alpha}(\overline{D})$. Then $u \in C^{2,\alpha}(\overline{D})$ if and only if

$$f(0,0) = f(1,0) = f(1,T) = f(0,T) = 0.$$

If in addition $f \in C^{1,\alpha}(\overline{D})$, then $u \in C^{3,\alpha}(\overline{D})$.

Moreover, following estimates on the solution of (2.1) holds [23];

Theorem 2.1. *There exists a number C, independent of perturbation parameter ϵ, such that for all sufficiently small positive values of ϵ the following relation holds:*

$$\|u(x,t) - \varphi(x)\| \leqslant CT, \text{ and}$$
$$\|u(x,t)\| \leqslant C', \quad t \in]0,T].$$

3. Time discretization

We consider the uniform time grid

$$T_k = \left\{ t_k : t_k = k\frac{T}{M} = k\Delta t,\ k = 0,1,\ldots M \right\}$$

and discretization of time variable using Euler's implicit rule yields