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On functional inequalities and their applications in the oscillation theory ☆

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ABSTRACT

In the paper, we present several functional inequalities and offer their application for higher order advanced differential equations of the form

$$x^{(n)}(t)+q(t)x(\sigma(t))=0,$$

to be oscillatory. The conditions obtained essentially improve, complements and extends many other known results and involves unimprovable constants.

1. Introduction

The paper is organized as follows. In the first part, we establish some functional inequalities between function and its derivatives and in the second part, we connect the estimate obtained with properties of solutions of advanced differential equations. So, we shall investigate properties of the functions $\sigma(t)\in C(I)$ and $x(t)\in C^{\ell}(I),\ I=(t_0,\infty)$.

Lemma 1. *Assume that $\ell\geqslant 2$ is a positive integer such that*

$$x(t)>0,\quad x'(t)>0,\ldots,\quad x^{(\ell)}(t)>0, \tag{C_ℓ}$$

eventually. Then for any constant $\lambda\in(0,1)$ and for all $i=1,2,\ldots,\ell-1$

$$\frac{t^i x^{(i)}(t)}{i!}\geqslant\lambda\binom{\ell-1}{i}x(t), \tag{1.1}$$

eventually.

Proof. Assume that (C_ℓ) holds for $t\geqslant t_0$. Using the fact that $x^{(\ell-2)}(t)\to\infty$ as $t\to\infty$ and the monotonicity of $x^{(\ell-1)}(t)$, it is easy to see that for any $k\in(0,1)$

$$kx^{(\ell-2)}(t)<x^{(\ell-2)}(t)-x^{(\ell-2)}(t_1)=\int_{t_1}^{t}x^{(\ell-1)}(s)\,ds\leqslant tx^{(\ell-1)}(t), \tag{1.2}$$

eventually, let us say, for $t\geqslant t_1\geqslant t_0$. For $\ell>2$, we define the sequence of functions as follows

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$$
\begin{aligned}
r_1(t) &= tx^{(\ell-1)}(t) - kx^{(\ell-2)}(t),\\
r_2(t) &= tx^{(\ell-2)}(t) - 2kx^{(\ell-3)}(t),\\
&\vdots\\
r_{\ell-1}(t) &= tx'(t) - (\ell-1)kx(t).
\end{aligned}
$$

It follows from (1.2) that $r_1(t) > 0$ and, moreover, $r_1(t) \to \infty$ as $t \to \infty$. On the other hand,

$$r_2'(t) \geqslant r_1(t) \to \infty \quad \text{as} \quad t \to \infty.$$

So we conclude that $r_2(t) \to \infty$ as $t \to \infty$, which implies $r_2(t) > 0$ eventually, let say for $t \geqslant t_2$. Repeating this procedure, we verify that $r_i(t) > 0$, eventually for all $i = 1, 2, \ldots, \ell-1$. Therefore,

$$
\begin{aligned}
tx'(t) &\geqslant (\ell-1)kx(t),\\
tx''(t) &\geqslant (\ell-2)kx'(t),\\
&\vdots\\
tx^{(\ell-1)}(t) &\geqslant kx^{(\ell-2)}(t),
\end{aligned}
$$

which yields

$$
\begin{aligned}
tx'(t) &\geqslant (\ell-1)kx(t),\\
t^2x''(t) &\geqslant (\ell-1)(\ell-2)k^2x(t),\\
&\vdots\\
t^{\ell-1}x^{(\ell-1)}(t) &\geqslant (\ell-1)!k^{\ell-1}x(t),
\end{aligned}
$$

Putting $\lambda = k^{\ell-1}$, the last inequalities imply (1.1) and the proof is complete. □

Now, we are prepared to prove desired functional estimates.

Lemma 2. *Assume that $\sigma(t) \geqslant t$ and that ℓ is a positive integer such that (C_ℓ) holds. Then for any constant $\lambda \in (0,1)$*

$$x(\sigma(t)) \geqslant \lambda\left(\frac{\sigma(t)}{t}\right)^{\ell-1}x(t), \tag{1.3}$$

eventually.

Proof. It follows from Taylor's theorem that

$$x(\sigma(t)) \geqslant x(t) + x'(t)(\sigma(t)-t) + \cdots + x^{(\ell-2)}(t)\frac{(\sigma(t)-t)^{\ell-2}}{(\ell-2)!} + x^{(\ell-1)}(t)\frac{(\sigma(t)-t)^{\ell-1}}{(\ell-1)!}.$$

Setting (1.1), one gets

$$x(\sigma(t)) \geqslant \lambda x(t)\left\{1 + \binom{\ell-1}{1}\left(\frac{\sigma(t)}{t}-1\right) + \binom{\ell-1}{2}\left(\frac{\sigma(t)}{t}-1\right)^2 + \cdots + \binom{\ell-1}{\ell-1}\left(\frac{\sigma(t)}{t}-1\right)^{\ell-1}\right\}.$$

Or in other words

$$x(\sigma(t)) \geqslant \lambda x(t)\left[1 + \left(\frac{\sigma(t)}{t}-1\right)\right]^{\ell-1} = \lambda\left(\frac{\sigma(t)}{t}\right)^{\ell-1}x(t).$$

The proof is complete. □

The obtained estimates can be used e.g. in the theory of functional equations. In the paper, we present their application in discussing oscillatory and asymptotic properties of higher order advanced differential equations.

2. Main results

We consider the higher order advanced differential equation

$$x^{(n)}(t) + p(t)x(\sigma(t)) = 0, \tag{E}$$

where

(H_1) $p(t) > 0, \sigma(t) \geqslant t$.