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Discretized Tikhonov regularization for Robin boundaries localization

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ABSTRACT

We deal with a boundary detection problem arising in nondestructive testing of materials. The problem consists in recovering an unknown portion of the boundary, where a Robin condition is satisfied, with the use of a Cauchy data pair collected on the accessible part of the boundary. We combine a linearization argument with a Tikhonov regularization approach for the local reconstruction of the unknown defect. Moreover, we discuss the regularization parameter choice by means of the so called balancing principle and we present some numerical tests that show the efficiency of our method.

1. Introduction

In this paper we deal with an inverse problem arising in corrosion detection. We consider a domain $\Omega \subset \mathbb{R}^2$ which models a 2*D* transverse section of a thin metallic specimen whose boundary is partly accessible and partly covered by a thin layer of an aggressive fluid. Hence, in order to detect the damage which is expected to occur in such a portion of the boundary, one has to rely on the electrostatic measurements of a potential u performed on the accessible portion.

The electrochemical phenomenon of surface corrosion in metals can be modeled by a boundary condition. The resulting equations, which have been studied by Vogelius and others [1–5], require a nonlinear boundary condition of the form

$$-\frac{\partial u}{\partial \nu}(x) = f(x,u), \tag{1.1}$$

where

$$f(x,u) = \lambda(x)(e^{\alpha(x)u(x)} - e^{(1-\alpha(x)u(x))}), \tag{1.2}$$

where $\lambda > 0$ is a parameter depending on the temperature and the ionic concentration of the fluid and $0 < \alpha < 1$ is a transfer coefficient.

However, as discussed in [6] a satisfactory approximation of (1.1) also for small scale problems can be constructed by first homogenizing the problem and then linearizing about the homogenized solution.

Such an approximation leads to the study of the following elliptic boundary value problem

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$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \Phi, & \text{on } \Gamma_A, \\ \frac{\partial u}{\partial \nu} + \gamma u = 0, & \text{on } \Gamma_I, \\ u = 0, & \text{on } \Gamma_D. \end{cases} \tag{1.3}$$

where $\gamma > 0$ is a constant.

According to this linear model, which was introduced by Inglese and Santosa [7] for γ not necessarily constant, u is the harmonic potential in the electrostatic conductor Ω, Φ is the prescribed current density on the portion Γ_A accessible to direct inspection. Whereas on Γ_I, the portion which is out of reach, the potential u satisfies a Robin boundary condition. In this paper we are interested in the numerical reconstruction issue of the unknown and damaged boundary Γ_I from the data collected on the accessible part of the boundary Γ_A, that is the Cauchy data pair $(u|_{\Gamma_A}, \Phi)$.

Boundary and parameter identification results related to this stationary inverse problem has been provided by many authors [8–21].

Local uniqueness and conditional stability results for the inverse problem at hand are contained in [11] and constitute the theoretical setting on which our numerical analysis relies. The present local determination of corroded boundaries consists in the localization of a small perturbation $\Gamma_{I,\theta}$ of a reference boundary Γ_I. It is convenient to introduce a small vector field $\theta \in C_0^1(\Gamma_I)$ so that the damaged domain Ω_θ is such that

$$\partial\Omega_\theta = \overline{\Gamma_A} \cup \overline{\Gamma_D} \cup \overline{\Gamma_{I,\theta}}, \quad \Gamma_{I,\theta} = \{z \in \mathbb{R}^2 : z = w + \theta(w), \quad w \in \Gamma_I\}.$$

Such a local approach combined with a linearization argument (see [11]) allows a reformulation of the problem of the localization of the unknown defect $\Gamma_{I,\theta}$ as the identification of the unknown term θ in a boundary condition of the type

$$\frac{\partial u'}{\partial \nu} + \gamma u' = \frac{\mathrm{d}}{\mathrm{d}s}\left(\theta \cdot \nu \frac{\mathrm{d}}{\mathrm{d}s} u\right) + \gamma\theta \cdot \nu(\gamma + 2H)u$$

at the portion Γ_I, where u' is a harmonic function satisfying homogeneous Neumann and Dirichlet conditions on Γ_A and Γ_D respectively, u is the solution of (1.3), and H denotes the curvature of the reference boundary Γ_I. As in [11] we carry over our analysis under the a priori assumption of a constant γ such that $2H(x) + \gamma > 0$ in Γ_I and we limit ourselves to the case of positive fluxes Φ only.

We linearize the forward map $F : \theta \mapsto u_\theta|_{\Gamma_A}$, where u_θ is the solution of the system (2.3) below, by its Fréchet derivative F' and take the *voltage contrast* on Γ_A, as the noisy right-hand term for the considered operator equation,

$$F'\theta = (u_\theta - u)|_{\Gamma_A}.$$

As in [17], we assume that the unavoidable measurement errors in *voltage contrast* are not smaller than the truncation error, $o(\|\theta\|)$. Therefore, if the noise level for voltage measurements is assumed to be δ, then the noise level for the right-hand term of the above operator equation can be written as $\tilde{\delta} = K\delta$, where a constant K is not necessary to be precisely known. Our method is based on a discretized Tikhonov regularization argument where the regularization parameter is chosen by a balancing principle (cf. [22–24]). Such an *a posteriori* parameter choice can lead to a regularized solution with order-optimal accuracy. At the same time it can provide a reliable estimate for the constant K.

2. Local identification of the unknown boundary

In this section we shall collect the main identifiability results of which our reconstruction procedure and our numerical tests are a follow up. For a more detailed description we refer to [11].

We denote with ν the outward normal to Γ_I and we assume that θ is a vector field in $C_0^1(\Gamma_I)$ having a nontrivial normal component θ_ν on Γ_I.

Let the Sobolev space $H_0^1(\Omega, \Gamma_D)$ be defined as follows

$$H_0^1(\Omega, \Gamma_D) = \{v \in H^1(\Omega) : \ v = 0 \text{ on } \Gamma_D \text{ in the trace sense}\}. \tag{2.1}$$

We introduce the forward map F

$$\begin{aligned} F : \quad & C_0^1(\Gamma_I) && \rightarrow && H^{\frac{1}{2}}(\Gamma_A), \\ & \theta && \mapsto && u_\theta|_{\Gamma_A}, \end{aligned} \tag{2.2}$$

where $u_\theta \in H_0^1(\Omega, \Gamma_D)$ is the solution to the elliptic problem

$$\begin{cases} \Delta u_\theta = 0 & \text{in } \Omega_\theta, \\ \frac{\partial u_\theta}{\partial \nu} = \Phi & \text{on } \Gamma_A, \\ \frac{\partial u_\theta}{\partial \nu} + \gamma u_\theta = 0 & \text{on } \Gamma_{I,\theta}, \\ u = 0 & \text{on } \Gamma_D. \end{cases} \tag{2.3}$$