Applied Mathematics and Computation 226 (2014) 455–483

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Asymptotically exact *a posteriori* LDG error estimates for one-dimensional transient convection–diffusion problems

Mahboub Baccouch

Department of Mathematics, University of Nebraska at Omaha, Omaha, NE 68182, United States

ARTICLE INFO

Keywords:
Local discontinuous Galerkin method
Transient convection–diffusion problems
Superconvergence
a posteriori error estimation
Radau points
Projections

ABSTRACT

In this paper, new *a posteriori* error estimates for the local discontinuous Galerkin (LDG) formulation applied to transient convection–diffusion problems in one space dimension are presented and analyzed. These error estimates are computationally simple and are computed by solving a local steady problem with no boundary conditions on each element. We first show that the leading error term on each element for the solution is proportional to a $(p+1)$-degree right Radau polynomial while the leading error term for the solution's derivative is proportional to a $(p+1)$-degree left Radau polynomial, when polynomials of degree at most p are used. These results are used to prove that, for smooth solutions, these error estimates at a fixed time converge to the true spatial errors in the L^2-norm under mesh refinement. More precisely, we prove that our LDG error estimates converge to the true spatial errors at $\mathcal{O}(h^{p+5/4})$ rate. Finally, we prove that the global effectivity indices in the L^2-norm converge to unity at $\mathcal{O}(h^{1/2})$ rate. Our computational results indicate that the observed numerical convergence rates are higher than the theoretical rates.

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1. Introduction

Problems involving convection and diffusion arise in several important applications throughout science and engineering, including fluid flow, heat transfer, etc. Their typical solutions exhibit boundary and/or interior layers. Designing and developing efficient, accurate, and robust numerical methods to address these problems has survived as a classical challenge since the earliest days of digital computation. The basic reasons for the difficulty involve (i) the development and tracking of discontinuities (for convection problems) or sharp transition layers (for convection–diffusion problems) and (ii) the generation of numerical solutions that fail to satisfy physical (e.g., entropy) principles. Classical numerical methods such as finite difference, finite volume, and finite element schemes have been developed to overcome these difficulties to a certain extent. However, they each suffer limitations that can potentially be overcome by the discontinuous Galerkin (DG) method, which combines the best features of each of the more traditional approaches. DG methods are becoming important techniques for the computational solution of many real-world problems. Many results for convection and diffusion equations indicate that DG methods provide effective ways to generate accurate error estimates.

In this paper we develop and analyze *a posteriori* error estimates of the spatial errors for the local DG (LDG) method applied to the transient convection–diffusion problems in one space dimension. The LDG finite element method is an extension of the DG method aimed at solving differential equations containing higher than first-order spatial derivatives. The DG method is a class of finite element methods using completely discontinuous piecewise polynomials for the numerical solution and the test functions. DG method combines many attractive features of the classical finite element and finite volume methods. It is a powerful tool for approximating some partial differential equations which model problems in physics, especially in fluid dynamics or electrodynamics. DG method was initially introduced by Reed and Hill in 1973 as a technique to solve neutron

E-mail address: mbaccouch@unomaha.edu

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http://dx.doi.org/10.1016/j.amc.2013.10.026

transport problems [55]. In 1974, LaSaint and Raviart [54] presented the first numerical analysis of the method for a linear advection equation. Since then, DG methods have been used to solve ordinary differential equations [7,29,52,54], hyperbolic [25–28,43,44,47,48] and diffusion and convection–diffusion [23,24,64,49] partial differential equations. Consult [42] and the references cited therein for a detailed discussion of the history of DG methods and a list of important citations on the DG method and its applications.

DG methods allow discontinuous bases, which simplify both h-refinement (mesh refinement and coarsening) and p-refinement (method order variation). However, for DG methods to be used in an adaptive framework one needs *a posteriori* error estimates to guide adaptivity and stop the refinement process. The solution space consists of piecewise continuous polynomial functions relative to a structured or unstructured mesh. As such, it can sharply capture solution discontinuities relative to the computational mesh. It maintains local conservation on an elemental basis. The success of the DG method is due to the following properties: (i) does not require continuity across element boundaries, (ii) is locally conservative, (iii) is well suited to solve problems on locally refined meshes with hanging nodes, (iv) exhibits strong superconvergence that can be used to estimate the discretization error, (v) has a simple communication pattern between elements with a common face that makes it useful for parallel computation and (vi) it can handle problems with complex geometries to high order.

The LDG method for solving convection–diffusion problems was first introduced by Cockburn and Shu in [45]. They further studied the stability and error estimates for the LDG method. Castillo et al. [31] presented the first *a priori* error analysis for the LDG method for a model elliptic problem. They considered arbitrary meshes with hanging nodes and elements of various shapes and studied general numerical fluxes. They showed that, for smooth solutions, the L^2 errors in ∇u and in u are of order p and $p + 1/2$, respectively, when polynomials of total degree not exceeding p are used. Cockburn et al. [40] presented a superconvergence result for the LDG method for a model elliptic problem on Cartesian grids. They identified a special numerical flux for which the L^2-norms of the gradient and the potential are of orders $p + 1/2$ and $p + 1$, respectively, when tensor product polynomials of degree at most p are used.

Recent work on other numerical methods for convection–diffusion and for pure diffusion problems has been reviewed by Cockburn et al. [41]. In particular, Baumann and Oden [24] presented a new numerical method which exhibits the best features of both finite volume and finite element techniques. Rivière and Wheeler [56] introduced and analyzed a locally conservative DG formulation for nonlinear parabolic equations. They derived optimal error estimates for the method. Rivière et al. [58] analyzed several versions of the Baumann and Oden method for elliptic problems. Wihler and Schwab [65] proved robust exponential rates of convergence of DG methods for stationary convection–diffusion problems in one space dimension. We also mention the work of Castillo, Cockburn, Houston, Süli, Schötzau and Schwab [59,32,33] in which optimal a priori error estimates for the *hp*-version of the LDG method for convection–diffusion problems are investigated. Later Adjerid et al. [8,9] investigated the superconvergence of the LDG method applied to diffusion and transient convection–diffusion problems. More recently, Celiker and Cockburn [34] proved a new superconvergence property of a large class of finite element methods for one-dimensional steady state convection–diffusion problems. Finally, we mention the recent work of Shu et al. [35,36] in which the superconvergence property of the LDG scheme for convection–diffusion equations in one space dimension are proven.

In recent years, the study of superconvergence and *a posteriori error* estimates of DG methods has been an active research field in numerical analysis, see the monographs by Verfürth [61], Wahlbin [63], and Babuška and Strouboulis [15]. A knowledge of superconvergence properties can be used to (i) construct simple and asymptotically exact *a posteriori* estimates of discretization errors and (ii) help detect discontinuities to find elements needing limiting, stabilization and/or refinement. Typically, *a posteriori* error estimators employ the known numerical solution to derive estimates of the actual solution errors. They are also used to steer adaptive schemes where either the mesh is locally refined (h-refinement) or the polynomial degree is raised (p-refinement). For an introduction to the subject of *a posteriori* error estimation see the monograph of Ainsworth and Oden [14]. Superconvergence properties for finite element and DG methods have been studied in [7,12,46,54,66,60] for ordinary differential equations, [3,16,7,10] for hyperbolic problems and [1,2,4,9–11,22,30,34,37,68] for diffusion and convection–diffusion problems. *A posteriori* error estimates for finite volume and mixed finite element methods for elliptic problems have been developed in [62,53,13]. Several *a posteriori* DG error estimates are known for hyperbolic [38,39,50] and diffusive [51,57] problems.

Related theoretical results in the literature including superconvergence results and error estimates of the LDG methods for convection–diffusion problems are given in [8,9,35,37,34,67]. Cheng and Shu [35] studied the convergence behavior of the LDG methods when applied to one-dimensional time dependent convection–diffusion equations. They observed that the LDG solution is superconvergent towards a particular projection of the exact solution. The order of superconvergence is observed to be $p + 2$ when polynomials of degree at most p are used. However, there is no theoretical justification of these results so far. In [37], Cheng and Shu studied the superconvergence property for the DG and LDG methods for solving one-dimensional time-dependent convection and convection–diffusion equations. They proved superconvergence towards a particular projection of the exact solutions. The order of superconvergence is proved to be $p + 3/2$ when p-degree piecewise polynomials with $p \geqslant 1$ are used. In [7], Adjerid et al. used the DG method to solve one-dimensional transient hyperbolic problems and showed that the local error on each element is proportional to a Radau polynomial. They further constructed implicit residual-based *a posteriori* error estimates but they did not prove their asymptotic exactness. Later, Adjerid and Baccouch [3,21] investigated the global convergence of the implicit residual-based *a posteriori* error estimates of Adjerid et al.

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