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Existence of travelling wave front solutions of a two-dimensional anisotropic model

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ABSTRACT

This paper considers a two-dimensional anisotropic model

$$\psi_t = r\psi - \frac{1}{k_0^4}\left(\Delta + k_0^2\right)^2\psi - \frac{\tilde{c}}{k_0^4}\partial_y^4\psi + \frac{2\eta}{k_0^4}\partial_x^2\partial_y^2\psi - \psi^3,$$

introduced by Pesch and Kramer (1986) [28]. Assume that ψ travels with a speed c in the propagation direction x and is periodic in the transverse direction y. This model is formulated as a spatial dynamic system in which the variable x is a time-like variable. A center-manifold reduction technique and a normal form analysis are applied to show that this dynamic system can be reduced to a system of ordinary differential equations. A bifurcation analysis yields the persistence of the heteroclinic orbit for the reduced system when higher order terms are added and the speed c is small enough, which establishes the existence of travelling wave front solutions. In order to overcome the difficulty caused by the irreversibility, some appropriate constants are adjusted.

1. Introduction

The structures and transitions in Rayleigh-Bénard convection of liquid layers heated from below [26] have received a lot of attention from the theoretical as well as experimental points of view over the past decades. Apart from the boundary effects this two-dimensional system is isotropic, which has numerous known results about the stationary structures such as straight rolls, squares or hexagons, grain boundaries and so on (see, e.g., [3,9,12,26,20]). Compared with the isotropic system, the anisotropic two-dimensional system with a preferred axis has attracted renewed interest in recent years [5,16,15,19,29,22,28] because of the presence of various different sorts of structures. For example, the normal rolls parallel or perpendicular to the preferred axis at the critical point can be found; the oblique rolls having an angle with respect to the preferred axis can occur; a rectangular structure parallel to the preferred axis can also be observed.

In this paper we consider a two-dimensional anisotropic model (also called generalized Swift–Hohenberg model). This model was introduced by Pesch and Kramer [28] in order to investigate a transition from normal to oblique roll-type structure, which is given by

$$\tau\psi_t = -\left(\lambda_1\partial_x^4 + \lambda_2\partial_y^4 + 2\lambda_3\partial_x^2\partial_y^2 + \mu_1\partial_x^2 + \mu_2\partial_y^2\right)\psi - \kappa\psi - \tilde{\gamma}\psi^3. \tag{1.1}$$

As [28] said, ψ can be interpreted as the small lateral displacement of a thin elastic plate extended in the (x, y)-plane, loaded along the x- and y-direction and embedded in an elastic medium. λ_i denotes the bending coefficients for $i = 1, 2, 3$. The

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loading is characterized by μ_1 and μ_2, and the linear and nonlinear restoring forces are described by κ and $\tilde{\gamma}$, respectively. After scaling, (1.1) can be written as (see [5 or 28] for $k_0 = 1$)

$$\psi_t = r\psi - \xi^4\left(\Delta + k_0^2\right)^2\psi - \frac{\tilde{c}}{k_0^4}\partial_y^4\psi + \frac{2\eta}{k_0^4}\partial_x^2\partial_y^2\psi - \psi^3, \tag{1.2}$$

where $k_0 > 0$ is the wave number of the base periodic pattern and

$$\xi = \frac{1}{k_0}, \quad \tilde{c} = \frac{\lambda_2\mu_1^2}{\lambda_1\mu_2^2} - 1, \quad \eta = 1 - \frac{\lambda_3\mu_1}{\lambda_1\mu_2}, \quad r = 1 - \frac{4\lambda_1\kappa}{\mu_1^2}. \tag{1.3}$$

(In (2.3) of [28], μ_2 in the expression of r should be μ_1.)

For $\tilde{c} = \eta = 0$, the above equation is reduced to the well-known Swift–Hohenberg (SH) model for studying nonlinear phenomena like granular media [2], elasticity and solid mechanics [10], chemical reactions [24], nonlinear optics [25], hydrodynamics [34] and so on. This model has a variational structure and it has drawn much attention in recent years. Most attempts have paid attention to its steady solutions, especially in one dimension. Van den Berg et al. [36] gave periodic solutions. Smets and Van den Berg [33] used the mountain-pass lemma and Struwe's monotonicity trick to obtain homoclinic orbits while Santra and Wei [31] got them by the energy and the Morse index. Breuer et al. [6] also studied them by a computer assisted proof. Bonheure [4] established the existence of multitransition kinks and pulses obtained as local minima of the associated functional. Applying the dynamic approach, Deng and Li [13] obtained a generalized homoclinic solution (a homoclinic orbit exponentially tending to a periodic solution). Van den Berg et al. [35] investigated its travelling wave fronts with the aid of the energy function and some conserved quantities. In addition, Haragus and Scheel considered the steady solutions in a stripe $(x, y) \in \mathbf{R} \times \mathbf{R}/(2\pi\mathbf{Z})$ with 2π-periodic boundary conditions in y, and got knee solutions, step solutions [18], and grain boundaries [20], which correspond to the reversible heteroclinic or homoclinic orbits of the reduced ordinary differential equations. In 2009, Burke et al. [8] pointed out that the spatial reversibility (i.e., the SH model is invariant under $R : x \to -x, \psi \to \psi$) plays a very important role in the study of its structures, and then they numerically discussed the SH mode with a small dispersive term, which breaks the reversibility.

For (1.2), Pesch and Kramer [28] derived envelope (or amplitude) equations for $k_0 = 1$ to show the possibility of a normal-oblique transition. Numerical study of its domain coarsening are also given in [5,15]. To our knowledge, there is no other results about this model and in particular no localized structures such as the travelling wave fronts [35] obtained in the SH model. Localized structures are very important in the pattern-formation community, and many recent experimental and theoretical studies have focused exclusively on localized patterns such as in chemical reactions [37], magnetoconvection [11], vegetation patches in deserts [32], surface structures in ferro-magnetic fluids [30], nonlinear optics [1], mathematical neuroscience and many others. Thus, the study of localized structures is of fundamental importance to research in all fields.

The objective of this paper is to obtain the existence of a travelling wave front solution of the Eq. (1.2) connecting two different periodic solutions (periodic only in the y-direction): this front corresponds to a heteroclinic orbit of the reduced ordinary differential equations of (1.2) (see Section 3). Assume that the solutions of (1.2) propagate in the x-direction with a small speed c and are periodic in the y-direction. Because it has no reversibility R, the study becomes much more complicated. Applying the methods given by Groves and Mielke [17] and Haragus and Scheel [18–20], we choose the variable x as a time-like variable and change it into a spatial dynamic system. The spectrum of the corresponding linearized operator consists entirely of isolated eigenvalues of finite algebraic multiplicity and contains infinite purely imaginary eigenvalues for $\tilde{c} \leqslant -1$. We study the case: there is only a double eigenvalue zero on the imaginary axis (If the double eigenvalues on the imaginary axis are not equal to zero, the problem becomes much more difficult because of the irreversibility and the higher dimensions, and it is left for our future work). Using a center-manifold reduction technique given by Mielke [27] and a normal form analysis, we show that this dynamic system can be reduced to a system of two-dimensional ordinary differential equations. Then a nonzero equilibrium is found, and a heteroclinic orbit of the normal form near this equilibrium is obtained. After a bifurcation analysis, we obtain that this heteroclinic orbit persists for $x \in [0, \infty)$ and $x \in (-\infty, 0]$ respectively when higher order terms are added and the speed c is small enough. Because of irreversibility of this system, we adjust some constants so that we can connect these two heteroclinic orbits to get a new heteroclinic orbit of the whole system for $x \in \mathbf{R}$, which establishes the existence of a travelling wave front solution.

The paper is organized as follows. Section 2 changes the Eq. (1.2) into a spatial dynamic system by using the methods from [17–20]. The properties and the adjoint operator of its linear operator are also stated. Section 3 studies the case: a double eigenvalue zero and other eigenvalues with nonzero real parts. The manifold reduction theorem [27] reduces the spatial dynamic system to a system of two ordinary differential equations. Its normal form is given. Section 4 gives the existence of a travelling wave front solution. The basic idea is similar to one in [17]. The normal form of the reduced system has a heteroclinic orbit. When the higher-order terms are included, a fixed point theorem and a perturbation method yield that this heteroclinic orbit persists after activating some constants. In Section 5, we give a short conclusion.

Notation 1. Throughout this paper, M denotes a positive constant and $B = O(C)$ means that $|B| \leqslant M|C|$.