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Willmore-like methods for the intersection of parametric (hyper)surfaces

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ABSTRACT

The aim of this paper is to obtain all Frenet apparatus of transversal intersection curves by giving new and easy applicable methods in three and four-dimensional Euclidean spaces. The methods includes a few operator which are obtained by using the idea of Willmore. By using these operators we obtain the Frenet vectors and curvatures of the transversal intersection curves of two (three) surfaces (hypersurfaces) in which at least one (hyper)surface is given parametrically.

1. Introduction

In differential geometry, the curvatures and Frenet vectors of a space curve which is given by its parametric equation can be found easily in Euclidean n-space $\mathbb{E}^n$ [see e.g. 1–7]. When we have a space curve given as an intersection curve in $\mathbb{E}^n$, it is hard to compute its Frenet apparatus. To overcome this difficulty, different methods have been given for transversal intersection curve of two surfaces in $\mathbb{E}^3$, and of three hypersurfaces in $\mathbb{E}^4$. Since (hyper)surfaces can be given by their parametric or implicit equations, the intersection problems occur as parametric-parametric, parametric-implicit, implicit-implicit in $\mathbb{E}^3$ and parametric-parametric-parametric, parametric-parametric-implicit, parametric-implicit-implicit, implicit-implicit-implicit in $\mathbb{E}^4$. Recently, these intersection problems have attracted much attention.

Hartmann gives formulas which compute the curvature of the intersection curves for all types of intersection problems in $\mathbb{E}^3$ [8]. Ye and Maekawa present algorithms for computing the curvatures, Frenet vectors and higher order derivatives of both transversal and tangential intersection curves of two surfaces for all types of intersection problems [9]. Goldman supplies curvature formulas for the intersection curve of two implicit surfaces and also the formula of the first curvature of the intersection curve of n implicit hypersurfaces [10]. Aléssio, using the implicit function theorem, obtains the Frenet apparatus of the intersection curve of two implicit surfaces in $\mathbb{E}^3$ [11], and of three implicit hypersurfaces in $\mathbb{E}^4$ [12]. Aléssio and Guadalupe present formulas for curvature, geodesic torsion and geodesic curvature for the intersection curve of two spacelike surfaces in the Lorentz–Minkowski 3-space [13]. Soliman et al. study the intersection curve of implicit and parametric surfaces [14]. Abdel-All et al. obtain the Frenet apparatus of the intersection curve of two implicit surfaces [15]. They also study the intersection curves of parametric-parametric-implicit and parametric-implicit-implicit hypersurfaces in $\mathbb{E}^4$ [16]. Düldül computes the Frenet apparatus of the transversal intersection curve of three parametric hypersurfaces in $\mathbb{E}^4$ [17].

Willmore considers the intersection curve of two implicit surfaces given by $f(x,y,z)=0$ and $g(x,y,z)=0$. By taking $\mathbf{h}=\nabla f\times\nabla g$ and defining the operator $\triangle=\lambda\frac{d}{ds}=\left(h_1\frac{\partial}{\partial x}+h_2\frac{\partial}{\partial y}+h_3\frac{\partial}{\partial z}\right)$, he obtains the curvature and the torsion of the intersection curve [18].

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Uyar Düldül and Düldül, extending the Willmore's above method into 4-space, obtain the Frenet apparatus of the transversal intersection curve of three implicit hypersurfaces [19]. Aléssio obtains the normal curvature, the first geodesic curvature and the first geodesic torsion for the transversal intersection curve of $n-1$ implicit hypersurfaces in $\mathbb{E}^n$ [20].

The simplicity of Willmore's method led us to look for similar ways for the transversal intersection problems

parametric-parametric, parametric-implicit in $\mathbb{E}^3$

and

parametric-parametric-parametric, parametric-parametric-implicit, parametric-implicit-implicit in $\mathbb{E}^4$.

In this paper, using the idea of Willmore, we provide new operators for these intersection problems. We thought that, since our new methods use only scalar and vector products, they can be easily applied to any intersection problem, and also can simplify the calculations for the above intersection problems compared with the given methods before. By using the given new operators, we study the differential geometry properties of transversal intersection curve of two surfaces in $\mathbb{E}^3$, and of three hypersurfaces in $\mathbb{E}^4$. For each case, we obtain the Frenet vectors $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ and $\{\mathbf{t},\mathbf{n},\mathbf{b}_1,\mathbf{b}_2\}$ and the curvatures $\{\kappa,\tau\}$ and $\{k_1,k_2,k_3\}$ in $\mathbb{E}^3$ and $\mathbb{E}^4$, respectively.

2. Preliminaries

2.1. Curves and surfaces in $\mathbb{E}^3$

Let $S\subset\mathbb{E}^3$ be a regular surface given by $\mathbf{X}=\mathbf{X}(u,v)$ and $\beta: I\subset\mathbb{R}\to S$ be an arbitrary curve with arc-length parametrization. If $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ is the moving Frenet frame along β, then the Frenet formulas are given by

$$\mathbf{t}'=\kappa\mathbf{n},\quad \mathbf{n}'=-\kappa\mathbf{t}+\tau\mathbf{b},\quad \mathbf{b}'=-\tau\mathbf{n}, \tag{2.1}$$

where $\mathbf{t},\mathbf{n},\mathbf{b}$ denote the tangent, the principal normal and the binormal vector fields; κ is the curvature and τ is the torsion of the curve β. Also, since S is regular, the partial derivatives $\mathbf{X}_u,\mathbf{X}_v$ are linearly independent at every point of S and the normal vector of S is given by $\mathbf{N}=\mathbf{X}_u\times\mathbf{X}_v$. Using the Frenet formulas, for the derivatives of the curve β, we have

$$\beta'=\mathbf{t},\quad \beta''=\mathbf{t}'=\kappa\mathbf{n},\quad \beta'''=-\kappa^2\mathbf{t}+\kappa'\mathbf{n}+\kappa\tau\mathbf{b}. \tag{2.2}$$

Since β lies on S, we may also write $\beta'=\mathbf{X}_u u'+\mathbf{X}_v v'$.

If the surface S is given by its implicit equation $f(x,y,z)=0$, then the surface normal is obtained by $\mathbf{N}=\nabla f=\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right)$.

2.2. Curves and surfaces in $\mathbb{E}^4$

Definition 2.1. The *ternary product* (or *vector product*) of the vectors $\mathbf{x}=\sum_{i=1}^4 x_i\mathbf{e_i}$, $\mathbf{y}=\sum_{i=1}^4 y_i\mathbf{e_i}$, and $\mathbf{z}=\sum_{i=1}^4 z_i\mathbf{e_i}$ is defined by [21,22]

$$\mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}=\begin{vmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4\\ x_1 & x_2 & x_3 & x_4\\ y_1 & y_2 & y_3 & y_4\\ z_1 & z_2 & z_3 & z_4\end{vmatrix},$$

where $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4\}$ denotes the standard basis of $\mathbb{E}^4$.

Let us consider a regular hypersurface $M\subset\mathbb{E}^4$ given by the parametric equation $\mathbf{R}=\mathbf{R}(u_1,u_2,u_3)$ and a unit speed curve $\alpha: I\subset\mathbb{R}\to M$. If $\{\mathbf{t},\mathbf{n},\mathbf{b}_1,\mathbf{b}_2\}$ denotes the moving Frenet frame along α, then the Frenet formulas are given by [23]

$$\mathbf{t}'=k_1\mathbf{n},\quad \mathbf{n}'=-k_1\mathbf{t}+k_2\mathbf{b}_1,\quad \mathbf{b}_1'=-k_2\mathbf{n}+k_3\mathbf{b}_2,\quad \mathbf{b}_2'=-k_3\mathbf{b}_1, \tag{2.3}$$

where $\mathbf{t},\mathbf{n},\mathbf{b}_1$, and $\mathbf{b}_2$ denote the tangent, the principal normal, the first binormal, and the second binormal vector fields; $k_i,\ (i=1,2,3)$ are the ith curvature functions of the curve α. The derivatives of the curve α is obtained as

$$\alpha'=\mathbf{t},\quad \alpha''=\mathbf{t}'=k_1\mathbf{n},\quad \alpha'''=-k_1^2\mathbf{t}+k_1'\mathbf{n}+k_1k_2\mathbf{b}_1, \tag{2.4}$$

$$\alpha^{(4)}=-3k_1k_1'\mathbf{t}+(-k_1^3+k_1''-k_1k_2^2)\mathbf{n}+(2k_1'k_2+k_1k_2')\mathbf{b}_1+k_1k_2k_3\mathbf{b}_2. \tag{2.5}$$

Besides, we may write $\alpha'(s)=\sum_{i=1}^3\mathbf{R}_i u_i'$, where $\mathbf{R}_i=\frac{\partial\mathbf{R}}{\partial u_i},\ i=1,2,3$.

The normal vector of M is given by $\mathbf{N}=\mathbf{R}_1\otimes\mathbf{R}_2\otimes\mathbf{R}_3$. If M is given by $g(x,y,z,w)=0$, then the normal vector is $\mathbf{N}=\nabla g=\left(\frac{\partial g}{\partial x},\frac{\partial g}{\partial y},\frac{\partial g}{\partial z},\frac{\partial g}{\partial w}\right)$.