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Optimal system, symmetry reductions and new closed form solutions for the geometric average Asian options

Zhiguo Wang [a], Luping Wang [a], Deng-Shan Wang [b], Yan Jin [c,*]

[a] School of Management, Xi'an Jiaotong University, Xi'an 710049, China
[b] School of Applied Science, Beijing Information Science and Technology University, Beijing 100192, China
[c] Economic and Management College of Wuhan University, Wuhan 430072, China

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ABSTRACT

In this paper, the Lie group analysis method is applied to the geometric average Asian option pricing equation in financial problems. Firstly, the complete Lie symmetry group and infinitesimal generators of this equation are derived. Then the optimal system with one parameter for the Lie symmetry algebra are obtained, which gives the possibility to describe a complete set of invariant solutions to the pricing equation. Finally, based on the optimal system the symmetry reductions and corresponding closed form solutions for the pricing equation are proposed.

1. Introduction

An Asian option is a special type of option contract, which is an averaging option whose terminal payoff is determined by the average underlying price over some pre-set period of time [1–13]. Because of this fact, Asian options have a lower volatility and hence render them cheaper relative to the usual European option and American option, where the payoff of the option contract depends on the price of the underlying instrument at exercise. They are commonly traded on currencies and commodity products which have low trading volumes. Asian options are so called because they were introduced in Tokyo, Japan, in 1987 when Banker's Trust Tokyo office used them for pricing average options on crude oil contracts. Thus Asian options are one of the basic forms of path-depending exotic options.

Asian options are broadly segregated into three categories: arithmetic average Asian options, geometric average Asian options and both these forms can be averaged on a weighted average basis. In general, there are not general explicit pricing formulae for the arithmetic averaged Asian options because the distribution of the arithmetic average of a set of lognormal distributions is not explicit. People think that the distribution of the geometric average of a set of lognormal distribution is also lognormal. So along this line Kemna and Vorst [6] and Conze and Viswanathan [7] obtain an explicit pricing formula for the Asian options on geometric average. Moreover, Bouaziz et al. [1] use a simple linearization procedure and propose an approximate closed-form solution to the pricing of "floating-strike" Asian options. The approximation suggested by Turnbull and Wakeman (TW) [8] makes use of the fact that the distribution under arithmetic averaging is approximately lognormal, and they put forward the first and second moments of the average in order to price the option. Caverhill and Clewlow [9] use the fast Fourier transform to obtain numerical approximations of the price of Asian options. Geman and Yor [10] propose an analytical study of Asian options. In particular, they characterize the case where an Asian call option price is higher than a standard European call option price. Alziary et al. [11] the Asian options analytically and numerical by a P.D.E. approach. Ju [12] produces an analytical approximation to price Asian options by assuming that even though the weighted average of

* Corresponding author.
E-mail address: jinyanwh2013@163.com (Y. Jin).

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lognormal variables is not lognormal, one can still be able to approximate the weighted average by a lognormal variable if the first two moments of moments are true. Most recently, Devreese et al. [13] derive a closed-form solution for the price of an average strike as well as an average price geometric Asian option, by making use of the path integral formulation.

There are two main classes of Asian options, the average price options and the average strike options. The corresponding terminal call payoff for the two classes are $\max\{J_T - K, 0\}$ and $\max\{S_T - J_T, 0\}$ for a call option, respectively. Here, S_T is the asset price at expiry, K is the strike price and J_T denotes some form of average of the price of the underlying asset over the averaging period $[0, T]$, either the arithmetical or geometrical average of the asset price. The value of J_T depends on the realization of the asset price path. The average price Asian options cost less than plain vanilla options, which are useful in protecting the owner from sudden short-lasting price changes in the market for example due to order imbalances.

The modern analysis of option pricing begins with the work of Black and Scholes [14] and Merton [15] in the early 1970s. The resulting model, called the Black–Scholes equation is a linear partial differential equation whose solution gives the fair price of a contingent claim. Under suitable assumptions, the Asian options can also be described by a linear partial differential equation [5]. There are two ways to solve the option pricing problems: numerical treatments [16–18] and analytical methods [19,18–21]. Recent years, there are an increasing number of researches about the use of symmetry analysis for the option pricing differential equations [20,21]. It is worth pointing out that the pioneering paper of Bordag and Chmakova [20] investigates the evaluation of an option hedge-cost under relaxation of the price-taking assumption by Lie group method. Specifically, they find some explicit solutions of the nonlinear Black–Scholes equation which incorporates the feedback-effect of a large trader in case of market illiquidity and show that these typical solutions would have a payoff which approximates a strangle. For a given differential equation, one first uses Lie point symmetry analysis to obtain its symmetry groups. Then under some mild conditions one can write down the reduced equation for the invariant solution with respect to a subgroup. However, there is almost always an infinite number of the subgroups so we need an optimal system to classify all possible group-invariant solutions to the option pricing differential equations [20,21]. The optimal systems and their corresponding group-invariant solutions have been discussed for a number of partial differential equations [20–24].

In the present paper, we construct the group-invariant optimal system of the Asian option pricing equation, from which the interesting exact closed form solutions are obtained. The paper is organized as follows. We present the geometric average Asian option pricing Black–Scholes equation in Section 2. In Section 3, the complete Lie symmetry group and infinitesimal generators of this pricing equation are derived, and the optimal system with one parameter for the Lie symmetry algebra is given. In Section 4, the similarity variables and closed form solutions of the pricing equation are obtained by using the optimal system. The conclusions are stated in Section 5.

2. Geometric average Asian options

In this section we derive the governing differential equation for the price of a geometric average Asian option using the Black–Scholes approach. We consider a call option contract that is written at time $\tau = 0$. The holder of the contract will have a right to claim the difference between the average rate J_T and a strike price K at maturity T, i.e., the payoff of this Asian call option is $\max\{J_T - K, 0\}$. The average rate J_T is determined by the geometric mean of the underlying asset price

$$J_n = \left(\prod_{i=1}^{n} S_{\tau_i}\right)^{\frac{1}{n}} = e^{\frac{1}{n}\sum_{i=1}^{n} \ln S_{\tau_i}} \tag{1}$$

for discrete geometric averaging and

$$J_\tau = e^{\frac{1}{\tau}\int_0^\tau \ln S_s ds} \tag{2}$$

for continuous geometric averaging, where S_{τ_i} is the asset price at discrete time τ_i.

Let $C(S, J, \tau)$ denote the value of the Asian option, which is a function of time and the two state variables, asset price S and average asset value J. Consider a portfolio that contains one unit of the Asian option and $-\Delta$ unit of the underlying asset. We then choose Δ such that the stochastic components associated with the option and the underlying asset cancel each other out. Assume the asset price dynamics to be given by

$$\frac{dS}{S} = \mu d\tau + \sigma dW, \tag{3}$$

where W is the standard Brownian process, μ and σ are the expected rate of return and volatility of the asset price, respectively. Let Π denote the value of the above portfolio, so the portfolio value is given by

$$\Pi = C(S, J, \tau) - \Delta S \tag{4}$$

and assuming Δ to be kept instantaneously "frozen". After considering the dividend yield q on the asset, the differential of Π can be found by Ito's formula [5] as

$$d\Pi = dC(S, J, \tau) - \Delta dS - q\Delta S d\tau = \left(\frac{\partial C}{\partial \tau} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - q\Delta S + \frac{\partial C}{\partial J}\frac{\partial J}{\partial \tau}\right)d\tau + \left(\frac{\partial C}{\partial S} - \Delta\right)dS, \tag{5}$$