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Polynomial solutions for a class of second-order linear differential equations

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ABSTRACT

We analyze the polynomial solutions of the linear differential equation $p_2(x)y''+p_1(x)y'+p_0(x)y=0$ where $p_j(x)$ is a jth-degree polynomial. We discuss all the possible polynomial solutions and their dependence on the parameters of the polynomials $p_j(x)$. Special cases are related to known differential equations of mathematical physics. Classes of new soluble problems are exhibited. General results are obtained for weight functions and orthogonality relations.

1. Introduction

The study of many physical phenomena can be reduced to the analysis of certain differential equations: these, in turn, demand insight for their solutions [1–12,28,34,38]. One of the simplest and most widely used linear differential equation is given [9,11,13–20,35] by

$$(a_{2,0}x^2+a_{2,1}x+a_{2,2})y''+(a_{1,0}x+a_{1,1})y'-\tau_{0,0}y=0,\qquad '\equiv d/dx, \tag{1}$$

where the parameters $a_{j,k}, j=0,1,2, k=0,1,2$ are real constants and $\tau_{0,0}$ is a function of some non-negative integer n. It is known that the differential equation (1) has an n-degree polynomial solution if

$$\tau_{0,0}=n(n-1)a_{2,0}+na_{1,0},\quad n=0,1,2,\ldots \tag{2}$$

The proof of this fact follows by substituting $y(x)=\sum_{k=0}^{n}c_kx^k$ in Eq. (1) and differentiating the resulting equation n times to obtain the condition (2). On the other hand, if a differential equation has the form

$$(a_{2,0}x^2+a_{2,1}x+a_{2,2})y''+(a_{1,0}x+a_{1,1})y'-[n(n-1)a_{2,0}+na_{1,0}]y=0, \tag{3}$$

where $a_{2,0}^2+a_{1,0}^2\neq 0$, then one solution of this differential equation is a polynomial of degree at most n. The proof of this claim is as follows. We first write Eq. (3) as

$$y''=\lambda_0(x)y'+s_0(x)y, \tag{4}$$

where

$$\lambda_0(x)=-\frac{(a_{1,0}x+a_{1,1})}{(a_{2,0}x^2+a_{2,1}x+a_{2,2})}\quad\text{and}\quad s_0(x)=\frac{n(n-1)a_{2,0}+na_{1,0}}{(a_{2,0}x^2+a_{2,1}x+a_{2,2})}. \tag{5}$$

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We then differentiate Eq. (4) $n-1$ and n times respectively to obtain

$$y^{(n+1)} = \lambda_{n-1}(x)y' + s_{n-1}(x)y \quad \text{and} \quad y^{(n+2)} = \lambda_n(x)y' + s_n(x)y, \tag{6}$$

where

$$\lambda_n(x) = \lambda'_{n-1}(x) + s_{n-1}(x) + \lambda_0(x)\lambda_{n-1}(x) \quad \text{and} \quad s_n(x) = s'_{n-1}(x) + s_0(x)\lambda_{n-1}(x), \quad n = 1,2,3,\ldots \tag{7}$$

Now multiply the first equation of (6) by $\lambda_{n-1}(x)$ and the second equation by $\lambda_n(x)$, then subtract the resulting equations to obtain

$$\lambda_{n-1}y^{(n+2)} - \lambda_n y^{(n+1)} = \delta_n y, \quad \delta_n = \lambda_n(x)s_{n-1}(x) - \lambda_{n-1}(x)s_n(x). \tag{8}$$

For $\lambda_0(x)$ and $s_0(x)$ given by (5), by induction we show that $\delta_n = 0$ for all $n = 1,2,\ldots$: For $n = 1$, a straightforward computation, using (5), allows us to conclude that $\delta_1 = 0$; assuming $\delta_n = 0$ we have for

$$\begin{aligned}
\delta_{n+1} &= \lambda_{n+1}s_n - \lambda_n s_{n+1} = (\lambda'_n + s_n + \lambda_0\lambda_n)s_n - \lambda_n(s'_n + s_0\lambda_n) = s_n^2\left[\left(\frac{\lambda_n}{s_n}\right)' + 1 + \lambda_0\left(\frac{\lambda_n}{s_n}\right) - s_0 b\left(\frac{\lambda_n}{s_n}\right)^2\right] \\
&= s_n^2\left[\left(\frac{\lambda_{n-1}}{s_{n-1}}\right)' + 1 + \lambda_0\left(\frac{\lambda_{n-1}}{s_{n-1}}\right) - s_0\left(\frac{\lambda_{n-1}}{s_{n-1}}\right)^2\right], \quad \text{since} \quad \delta_n = 0 \\
&= s_n^2\left[\frac{\lambda'_{n-1}}{s_{n-1}} - \frac{\lambda_{n-1}s'_{n-1}}{s_{n-1}^2} + 1 + \lambda_0\left(\frac{\lambda_{n-1}}{s_{n-1}}\right) - s_0\left(\frac{\lambda_{n-1}}{s_{n-1}}\right)^2\right] \\
&= s_n^2\left[\frac{\lambda'_{n-1} + s_{n-1} + \lambda_0\lambda_{n-1}}{s_{n-1}} - \frac{\lambda_{n-1}(s'_{n-1} + s_0\lambda_{n-1})}{s_{n-1}^2}\right] = s_n^2\left[\frac{\lambda_n}{s_{n-1}} - \frac{s_n\lambda_{n-1}}{s_{n-1}^2}\right] = \frac{s_n^2}{s_{n-1}^2}\cdot\delta_n = 0.
\end{aligned}$$

Eq. (8) then implies

$$y^{(n+1)} = C_1 \exp\left(\int^x \frac{\lambda_{n-1}(\tau)}{\lambda_n(\tau)}d\tau\right), \tag{9}$$

where C_1 is a constant. Thus, Eq. (6) reduces to a first-order linear equation with a solution of the form

$$y(x) = \exp\left(-\int^x \frac{s_{n-1}(\tau)}{\lambda_{n-1}(\tau)}d\tau\right). \tag{10}$$

If we differentiate (10) and substitute $y' = -s_{n-1}(x)\cdot y/\lambda_{n-1}(x)$ in Eq. (6) once again, we obtain $y^{(n+1)}(x) = 0$, i.e. $y(x)$ is a polynomial of degree at most n.

The differential equation (3) represents the source of many well-established results in the realm of special functions and orthogonal polynomials [3,9,11,13–19,21,22]. The classification of the standard orthogonal polynomials, for example, follows from the analysis of the polynomial solutions of this differential equation [9,14]. These results have found very many applications in diverse areas of mathematical and theoretical physics. For illustrative purposes we cite only a few references here: [1,2,4–8,10,12,23,25–27,29–33,36,37]. Most of the results concerning the polynomial solutions of Eq. (1) are obtained either as special cases of the parameters $a_{j,k}$ or through various transformations to reduce the complexity of dealing with differential equation directly. For example, for $a_{2,0} \neq 0$, with

$$p = \mp\sqrt{\frac{a_{2,1}^2 - 4a_{2,0}a_{2,2}}{a_{2,0}}}, \quad q = \frac{-a_{2,1} \pm \sqrt{a_{2,1}^2 - 4a_{2,0}a_{2,2}}}{2a_{2,0}}$$

the change of variable $x = pt + q$ reduces the differential equation (3) to the classical hypergeometric differential equation

$$t(1-t)\frac{d^2y}{dt^2} \frac{-(a_{1,0}(pt+q) + a_{1,1})}{pa_{2,0}}\frac{dy}{dt} + \frac{n(n-1)a_{2,0} + na_{1,0}}{a_{2,0}}y = 0 \tag{11}$$

and with

$$p = -\frac{\sqrt{a_{2,1}^2 - 4a_{2,0}a_{2,2}}}{2a_{2,0}}, \quad q = -\frac{a_{2,1}}{2a_{2,0}} \tag{12}$$

the differential equation (3) reduces to

$$(1-t^2)\frac{d^2y}{dt^2} - \frac{(a_{1,0}(pt+q) + a_{1,1})}{pa_{2,0}}\frac{dy}{dt} + \frac{n(n-1)a_{2,0} + na_{1,0}}{a_{2,0}}y = 0. \tag{13}$$

However, such transformations do not always allow us to study all the possible cases. For example, the case of $a_{2,1}^2 - 4a_{2,0}a_{2,2} \leqslant 0$ or $a_{2,0} = 0$. The present work focuses on studying the polynomial solutions of Eq. (3) directly, without the application of any transformation or reduction process. In other words, we study the polynomial solutions in terms of the parameters $a_{j,k}$ themselves. To our knowledge, no such study is available in the vast literature on the subject. In Section 2, we present the most gen-