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Existence of positive solutions for a class of nonlinear fractional differential equations with integral boundary conditions and a parameter

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ABSTRACT

In this paper, we study the existence of positive solutions for the following nonlinear fractional differential equations with integral boundary conditions:

$$\begin{cases} D_{0+}^{\alpha}u(t)+h(t)f(t,u(t))=0, \quad 0<t<1, \\ u(0)=u'(0)=u''(0)=0, \\ u(1)=\lambda\int_0^{\eta}u(s)\mathrm{d}s, \end{cases}$$

where $3<\alpha\leq 4, 0<\eta\leq 1, 0\leqslant\frac{\lambda\eta^{\alpha}}{\alpha}<1, D_{0+}^{\alpha}$ is the standard Riemann–Liouville derivative. $h(t)$ is allowed to be singular at $t=0$ and $t=1$. By using the properties of the Green function, u_0-bounded function and the fixed point index theory under some conditions concerning the first eigenvalue with respect to the relevant linear operator, we obtain some existence results of positive solution.

1. Introduction

Fractional differential equations have been of great interest recently. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications of such constructions in various fields of sciences and engineering such as control, porous media, electromagnetic, and other fields (see [1–8,10–12] and the references therein).

In present, most papers are devoted to the sublinear problem, see [2,3]. However, there are few papers consider the superlinear problem, see [1]. In this paper, we consider the following nonlinear fractional differential equations with integral boundary conditions

$$\begin{cases} D_{0+}^{\alpha}u(t)+h(t)f(t,u(t))=0, \quad 0<t<1, \\ u(0)=u'(0)=u''(0)=0, \\ u(1)=\lambda\int_0^{\eta}u(s)\mathrm{d}s, \end{cases} \tag{1}$$

under sublinear and superlinear cases, where $3<\alpha\leq 4, 0<\eta\leq 1, 0\leqslant\frac{\lambda\eta^{\alpha}}{\alpha}<1, D_{0+}^{\alpha}$ is the standard Riemann–Liouville derivative. The existence and multiplicity of positive solutions are obtained by means of the properties of the Green function, u_0-bounded function and the fixed point index theory under some conditions concerning the first eigenvalue with respect to the relevant linear operator. The methods are different from those in previous works. A function $u\in C^3([0,1],\mathbb{R}^+)\cap C^4((0,1),\mathbb{R}^+)$ is called a positive solution of FBVP (1) if it satisfies (1).

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The paper is organized as follows. Firstly, we derive the corresponding Green's function known as fractional Green's function and argue its positivity. This is our main work in this paper. Consequently, BVP (1) is reduced to an equivalent Fredholm integral equation. Finally, the existence results of positive solutions are obtained by the use of fixed point index and spectral radii of some related linear integral operators.

2. Background materials and Green's function

For the convenience of the reader, we present here the necessary definitions from fractional calculus theory to facilitate analysis of BVP (1). These definitions can be found in the recent literature, see [1–4]. Let $E = C[0,1]$ be the Banach space with the maximum norm $\|u\| = \max_{t\in[0,1]} |u(t)|$.

Definition 2.1. The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $y : (0,\infty) \to R$ is given by

$$I^{\alpha}_{0+}y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\mathrm{d}s,$$

provided the right-hand side is pointwise defined on $(0,\infty)$.

Definition 2.2. The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a continuous function $y : (0,\infty) \to R$ is given by

$$D^{\alpha}_{0+}y(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}}\mathrm{d}s,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number α, provided that the right-hand side is pointwise defined on $(0,\infty)$.

From the definition of the Riemann–Liouville derivative, we can obtain the statement.

Lemma 2.1 [2]. *Let $\alpha > 0$. If we assume $u \in C(0,1) \cap L^1(0,1)$, then the fractional differential equation*

$$D^{\alpha}_{0+}u(t) = 0,$$

$u(t) = C_1t^{\alpha-1} + C_2t^{\alpha-2} + \cdots + C_Nt^{\alpha-N}, C_i \in R(i = 1,2,\ldots,N)$, as unique solutions, where N is the smallest integer greater than or equal to α.

Lemma 2.2 [2]. *Assume that $u \in C(0,1) \cap L^1(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L^1(0,1)$. Then*

$$I^{\alpha}_{0+}D^{\alpha}_{0+}u(t) = u(t) + C_1t^{\alpha-1} + C_2t^{\alpha-2} + \cdots + C_Nt^{\alpha-N},$$

for some $C_i \in R(i = 1,2,\ldots,N)$, where N is the smallest integer greater than or equal to α.

In the following, we present Green's function of the fractional differential equation boundary value problem.

Lemma 2.3. *Given $y \in C(0,1) \cap L^1(0,1)$. The problem*

$$\begin{cases} D^{\alpha}_{0+}u(t) + y(t) = 0, \\ u(0) = u'(0) = u''(0) = 0, \\ u(1) = \lambda\int_0^{\eta} u(s)\mathrm{d}s, \end{cases} \tag{2}$$

where $0 < t < 1, 3 < \alpha \le 4, 0 < \eta \le 1, 0 \leqslant \frac{\lambda\eta^{\alpha}}{\alpha} < 1$, is equivalent to

$$u(t) = \int_0^1 G(t,s)y(s)\mathrm{d}s,$$

where

$$G(t,s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-1} - \frac{\lambda}{\alpha}(\eta-s)^{\alpha}t^{\alpha-1} - (1-\frac{\lambda}{\alpha}\eta^{\alpha})(t-s)^{\alpha-1}}{p(0)\Gamma(\alpha)}, & 0 \leqslant s \leqslant t \le 1,\ s \leqslant \eta; \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1} - (1-\frac{\lambda}{\alpha}\eta^{\alpha})(t-s)^{\alpha-1}}{p(0)\Gamma(\alpha)}, & 0 \leqslant \eta \leqslant s \leqslant t \le 1; \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1} - \frac{\lambda}{\alpha}(\eta-s)^{\alpha}t^{\alpha-1}}{p(0)\Gamma(\alpha)}, & 0 \leqslant t \leqslant s \leqslant \eta \le 1; \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{p(0)\Gamma(\alpha)}, & 0 \leqslant t \leqslant s \le 1,\ \eta \leqslant s, \end{cases} \tag{3}$$

here, $p(s) := 1 - \frac{\lambda\eta^{\alpha}}{\alpha}(1-s)$, $G(t,s)$ is called the Green function of BVP (2). Obviously, $G(t,s)$ is continuous on $[0,1] \times [0,1]$.