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Least squares solutions of the matrix equation $AXB + CYD = E$ with the least norm for symmetric arrowhead matrices ☆

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ABSTRACT

In this paper, the least squares solution of the matrix equation $AXB + CYD = E$ for symmetric arrowhead matrices with the least norm is discussed. By using Moore–Penrose inverse and the Kronecker product, the general expression of the solution to this problem is derived. A corresponding numerical algorithm and an example are also given.

1. Introduction and preliminaries

In this paper the following notations are used. Let $\mathbb{R}^{m\times n}$ be the set of all $m\times n$ real matrices, $\mathbb{OR}^{n\times n}$ be the set of all $n\times n$ orthogonal matrices, and I_n be the identity matrix of order n. We denote by A^+ and $\|A\|$ the Moore–Penrose inverse and the Frobenius norm of a real matrix A, respectively. For $A, B \in \mathbb{R}^{m\times n}$, let $(A, B) := tr(B^TA)$ denote the inner product of matrices A and B. Therefore, $\mathbb{R}^{m\times n}$ is a complete inner product space and the norm of a matrix generated by the inner product is the Frobenius norm, i.e. $\|A\| =< A, A>^{1/2}$. Let $A = (a_{ij})_{m\times n}, B \in \mathbb{R}^{p\times q}$. We denote by $A \otimes B = (a_{ij}B) \in \mathbb{R}^{mp\times nq}$ the Kronecker product of A and B. And $vec(A)$ stands for the vector $(a_{11}, a_{21}, \ldots, a_{m1}, a_{12}, a_{22}, \ldots, a_{m2}, \ldots, a_{1n}, a_{2n}, \ldots, a_{mn})^T$.

Definition 1.1 [1]. Let $A \in \mathbb{R}^{n\times n}$. A is called the symmetric arrowhead matrices if it has the following form:

$$A = \begin{bmatrix} a_1 & b_1 & b_2 & \cdots & b_{n-1} \\ b_1 & a_2 & 0 & \cdots & 0 \\ b_2 & 0 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n-1} & 0 & 0 & \cdots & a_n \end{bmatrix}.$$

and $vec_s(A)$ stands for the corresponding vector $(a_1, b_1, \ldots, b_{n-1}, a_2, \ldots, a_n)^T$.

We denote all symmetric $n\times n$ arrowhead matrices by $\mathbb{SAR}^{n\times n}$.

Such matrices arises in the description of radiationless transitions in isolated molecules and of oscillators vibrationally coupled with a Fermi liquid [2]. In modern control theory, symmetric arrowhead matrices could represent the parameter matrices of in the control equations of nonlinear control systems [2]. Recent developments in electromagnetic compatibility have also predicted potential applications of symmetric arrowhead matrices in the mathematical representation of electromagnetic interference factors.

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Considering the following matrix equation

$$AXB + CYD = E, \tag{1.1}$$

many studies have been done on the solution and the corresponding approximation problems of this matrix equation. In 1987, Chu studied the compatibility of Eq. (1.1) by using the general singular value decomposition (GSVD), and gave the least norm solution when the solution exists [3]. In 1998, Xu and Zheng obtained the least squares solution and the symmetric (anti-symmetric) solution of $AXA^H + CYC^H = F$ by using the canonical correlation decomposition (CCD) [4]. In 2006, Liao, Bai and Lei studied the least squares solution of $AXB^H + CYD^H = E$ with the least norm by combining CCD and GSVD [5]. The corresponding numerical algorithms are also proposed [8–11].

In this paper, we mainly consider the least-square solutions to matrix equation (1.1) for arrow-head matrices, which is described as follows.

Problem 1. Given $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times s}$, $C \in \mathbb{R}^{m\times k}$, $D \in \mathbb{R}^{k\times s}$, $E \in \mathbb{R}^{m\times s}$, and let

$$S_E = \left\{[X,Y] | X \in \mathbb{SAR}^{n\times n}, Y \in \mathbb{SAR}^{k\times k}, \|AXB + CYD - E\| = \min\right\}.$$

Finding out $[\widehat{X}, \widehat{Y}] \in S_E$ such that

$$\|\widehat{X}\|^2 + \|\widehat{Y}\|^2 = \min_{[X,Y]\in S_E} (\|X\|^2 + \|Y\|^2). \tag{1.2}$$

Our motivation are twofold: (1) The least-square solutions to matrix equation (1.1) have its own importance and applications. For instance, it is essential to the inverse scattering problem for the Helmholtz equation [12]. And the arrowhead matrix is an important class of matrices with many applications in linear modeling and control theory (see [2] for more references). The least-square solutions to (1.1) for symmetric and asymmetric matrices have been discussed [3–5], and this article could extend the results for arrow head matrices by solving Problem 1. (2) The least-square solutions to (1.1) for symmetric arrowhead matrices have several potential applications. For example, the generalized inverse eigenvalue problem (see [13,14] for details) can be reformulated as $AX = \Lambda BX$, where all column vectors of X are given eigenvectors, Λ is a diagonal matrix with each diagonal element being the given eigenvalue, and A, B are two symmetric arrowhead matrices to be determined. If the inverse eigenvalue problem is inconsistent, i.e., there exist no A, B such that $AX = \Lambda BX$, then we can obtain the best approximations by solving Problem 1.

To solve Problem 1, we utilize the Kronecker product and the Moore–Penrose inverse, based on which, the expression of S_E is obtained and the explicit formula of the solution of Problem 1 is derived.

This article is organized as follows. In Section 2, we give the main results of this paper. In Section 3, we propose an algorithm based on theorems in Section 2, and demonstrate the algorithm by a numerical example.

2. Solutions of Problem 1

In this section, we will discuss the solution of Problem 1. We begin with some lemmas.

Lemma 2.1. *Let $X \in \mathbb{SAR}^{n\times n}$, then $vec(X) = H_n\, vec_s(X)$, where*

$$H_n = \begin{bmatrix} e_1 & e_2 & e_3 & \cdots & e_{n-1} & e_n & 0 & 0 & \cdots & 0 & 0 \\ 0 & e_1 & 0 & \cdots & 0 & 0 & e_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & e_1 & \cdots & 0 & 0 & 0 & e_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & e_1 & 0 & 0 & 0 & \cdots & e_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & e_1 & 0 & 0 & \cdots & 0 & e_n \end{bmatrix}, \tag{2.1}$$

and e_i is a unit vector with the ith element one and the others zero.

Proof. If $X \in \mathbb{SAR}^{n\times n}$, form Definition 1.1, we have

$$X = \begin{bmatrix} x_{11} & x_{21} & x_{31} & \cdots & x_{n1} \\ x_{21} & x_{22} & 0 & \cdots & 0 \\ x_{31} & 0 & x_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n1} & 0 & 0 & \cdots & x_{nn} \end{bmatrix} = x_{11}(e_1, 0, \ldots, 0, 0) + x_{21}(e_2, e_1, \ldots, 0, 0)$$
$$+ \cdots + x_{(n-1)1}(e_{n-1}, 0, \ldots, e_1, 0) + x_{n1}(e_n, 0, \ldots, 0, e_1) + x_{22}(0, e_2, 0, \ldots, 0) + x_{33}(0, 0, e_3, \ldots, 0) + \cdots + x_{nn}(0, 0, \ldots, 0, e_n).$$